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AdS spacetimes from wrapped M5 branes

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Abstract: We derive a complete geometrical characterisation of a large class of AdS_3, AdS_4 and AdS_5 supersymmetric spacetimes in eleven-dimensional supergravity using G-structures. These are obtained as special cases of a class of supersymmetric $\mathbb{R}^{1,1}$, $\mathbb{R}^{1,2}$ and $\mathbb{R}^{1,3}$ geometries, naturally associated to M5-branes wrapping calibrated cycles in manifolds with G_2, SU(3) or SU(2) holonomy. Specifically, the latter class is defined by requiring that the Killing spinors satisfy the same set of projection conditions as for wrapped probe branes, and that there is no electric flux. We show how the R-symmetries of the dual field theories appear as isometries of the general AdS geometries. We also show how known solutions previously constructed in gauged supergravity satisfy our more general G-structure conditions, demonstrate that our conditions for half-BPS AdS_5 geometries are precisely those of Lin, Lunin and Maldacena, and construct some new singular solutions.

Keywords: Differential and Algebraic Geometry, M-Theory, AdS-CFT Correspondence.

Contents

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1. Introduction

M-theory on a supersymmetric background that contains an AdS_{d+2} factor is expected to be dual to a $d+1$-dimensional superconformal field theory (SCFT) [1]. A key issue is thus both to characterise the geometry of the generic eleven-dimensional supergravity backgrounds of this type and to find explicit new examples. In this paper we present a relatively simple way to describe a large class of such spacetimes in terms of G-structures, and show how some known solutions fit into this framework. We will also present some new but singular solutions.

A number of results characterising AdS solutions in M-theory have already appeared in the literature. The generic minimally supersymmetric backgrounds with an AdS_3 factor was analysed in [2] (see also [3]). Various authors have considered minimal AdS_4 compactifications; a general analysis is carried out in [4] and this is extended in [5] (although our results differ slightly from those in [4]). The generic AdS_5 case, dual to $\mathcal{N}=1$ SCFTs, was analysed in [6], while the AdS_5 case dual to $\mathcal{N}=2$ SCFTs was analysed in [7].

In this paper we will focus on AdS solutions with no electric flux. While this eliminates, for example, AdS_4 solutions of Freund-Rubin type, it still includes rich classes of known solutions. One class was originally derived from a gauged supergravity analysis. This work began with the two AdS_5 solutions of [8] dual to $\mathcal{N}=1$ and $\mathcal{N}=2$ SCFTs, and corresponding to M5-branes wrapping holomorphic two-cycles in Calabi-Yau three-folds and two-folds, respectively. This construction was extended to AdS solutions corresponding to M5-branes wrapping various supersymmetric cycles in [9–11] and including additional AdS_3 and AdS_4 examples with vanishing electric flux (for a review see [12]). A second class arose from the general AdS_5 analysis of [6]. By assuming that the compact internal six-manifold is complex, an infinite number of new explicit solutions were found. The six-manifolds were all S^2 bundles over a Kähler-Einstein four manifold with positive curvature, or over a product $S^2\times S^2$, $S^2\times T^2$, $S^2\times H^2$. One specific example in the $S^2\times H^2$ class gives the $\mathcal{N}=1$ AdS_5 solution of [8].

It is natural, therefore, to (i) characterise AdS geometries preserving various amounts of supersymmetry, (ii) recover the known wrapped brane solutions, (iii) attempt to generalise them to new, perhaps infinite, classes of solutions. In this paper we have achieved (i) and (ii), and have initiated the analysis of (iii), by reducing the general G-structure conditions to systems of first order ODEs, for generalisations of known gauged supergravity solutions, and also for some other special cases. While we have not found any new regular solutions, physically one would expect many new solutions, dual to SCFTs arising from wrapped branes, to be found in the classes that we analyse, but we are agnostic as to whether or not they can be found in explicit form.

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Of course, from a more general perspective, having a general characterisation of the AdS geometries is the first step in developing existence theorems, which is a longer term goal. In addition, the supersymmetric AdS geometries we characterise here will inherit rich structures from the dual SCFT, analogous to those elucidated for the type IIB $D = 5$ Sasaki-Einstein case in the beautiful work of [13, 14]. It is also worth noting that after analytic continuation the geometries we study should also describe supersymmetric excitations of $AdS_{d+2} \times S^{9-d}$ geometries, analogous to the "bubbling spacetimes" discussed in [7].

Our approach is motivated by the general analysis of the supersymmetric AdS_5 geometries in [6]. This was carried out by directly studying the canonical G-structure specified by the Killing spinors. However, it was also shown that the general result could be obtained by an alternative strategy and this is the one that we shall employ here. Using Poincaré coordinates, one views the AdS_{d+2} solution as a special case of an $\mathbb{R}^{1,d}$ solution. If one has a characterisation of the most general supersymmetric $\mathbb{R}^{1,d}$ solution, one can therefore extract the conditions for the most general AdS_{d+2} solution as a special case. In fact, to obtain the most general AdS solution, it turns out that it is not always necessary to start with the most general supersymmetric $\mathbb{R}^{1,d}$ solution.

For the case of AdS_5 it is sufficient to start with supersymmetric $\mathbb{R}^{1,3}$ geometries in a special class that we will call "wrapped brane geometries". These geometries are characterised by the fact that the Killing spinors are proportional to those preserved by a probe M5-brane wrapping a supersymmetric cycle in a special holonomy manifold: in particular they satisfy the same algebraic conditions.[1] We emphasise that the wrapped brane geometries need not contain branes. We shall explain this in more detail in section 2. Following this we will present the necessary and sufficient conditions for various wrapped brane geometries with $\mathbb{R}^{1,1}$, $\mathbb{R}^{1,2}$ and $\mathbb{R}^{1,3}$ factors preserving various amounts of supersymmetry. Using these results we are then able to characterise the supersymmetric AdS geometries.

At this point it is perhaps helpful to draw an analogy with supersymmetric type IIB $AdS_5 \times X_5$ solutions where X_5 is Sasaki–Einstein. This class of solutions can be derived from the more general class of supersymmetric solutions describing D3-branes that are transverse to a Calabi–Yau three-fold. Specifically, one obtains $AdS_5 \times X_5$ in the special case that the Calabi–Yau is a singular cone. This perspective plays a crucial role in identifying the dual SCFTs which live on the D3-branes. The wrapped M5 brane $\mathbb{R}^{1,d}$ solutions that we analyse here are the analogue of the general IIB solutions describing D3-branes transverse to the Calabi-Yau three-fold. Taking the special case which gives an AdS_{d+2} factor is the analogue of requiring that the Calabi–Yau three-fold be a cone. We similarly expect that, ultimately, this perspective will be key in understanding the dual SCFTs as the decoupling limit of some brane configuration. It is also worth noting that our approach can equally be used to analyse supersymmetric AdS solutions of any supergravity theory.

A natural question to ask is whether the generic supersymmetric AdS backgrounds can all be reproduced from the corresponding wrapped-brane geometries. As we mentioned above this is certainly true for the minimally supersymmetric AdS_5 case [6]. We will see

[1]These algebraic conditions can be equivalently phrased in terms of intersecting brane configurations.

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that it is also true for minimally supersymmetric AdS_4 spacetimes with vanishing electric flux, and very likely to be true for AdS_5 spacetimes with additional supersymmetry. This at least suggests that the wrapped-brane subclass is sufficiently general to give the generic AdS backgrounds with vanishing electric flux for all the cases we consider. We will return to this question in our conclusions.

The plan of the rest of this paper is as follows. In section 2, we give a general discussion of the wrapped brane configurations that we consider, together with an overview of their G-structures. In section 3, we give the necessary and sufficient conditions for supersymmetry of these classes of wrapped brane solutions. These conditions can be neatly understood using the generalised calibration conditions of [15] and this is discussed in section 4.

In section 5, we describe how we take the AdS limit of the wrapped brane metrics and in sections 6–8 we consider various cases. Section 6 analyses AdS_3 solutions that are dual to two-dimensional SCFTs with $\mathcal{N} = (2,0)$ and $\mathcal{N} = (4,0)$ supersymmetry, section 7 analyses AdS_4 solutions that are dual to three-dimensional SCFTs with $\mathcal{N} = 1$ and 2 supersymmetry, and section 8 analyses AdS_5 solutions that are dual to four-dimensional SCFTs with $\mathcal{N} = 2$ supersymmetry (the ones with $\mathcal{N} = 1$ supersymmetry are analysed in [6]).

Section 9 discusses explicit examples of solutions of the AdS supersymmetry conditions. We provide a general discussion of the G-structures underlying gauged supergravity AdS solutions, and show explicitly how they are realised for known examples. We highlight potential generalisations of the gauged supergravity solutions, discuss several other types of solutions, and explicitly construct new singular AdS_3 solutions arising from branes wrapping Kähler four-cycles in Calabi-Yau three-folds.

Section 10 concludes. We have relegated some technical material from the main text to several appendices. In appendix A we have listed the spinor projections used to define the wrapped M5-brane geometries and also the corresponding G-structures. Appendix B gives more of the general technical details involved in taking the AdS limit of the wrapped brane configurations. In appendix C, we give a representative example of the derivation of the AdS supersymmetry conditions from the wrapped brane supersymmetry conditions. In appendix D, we prove that the supersymmetry conditions for an M5 wrapping a Kähler two-cycle in a Calabi–Yau two-fold descend, in the AdS_5 limit, precisely to the half-BPS AdS_5 conditions of [7]. We will use the conventions of [16] throughout this paper.

2. Wrapped-brane spacetimes

The main objective of the paper is to characterise supersymmetric AdS_{d+2} geometries that come from wrapped M5-brane geometries. The wrapped brane geometries contain $\mathbb{R}^{1,d}$ factors and have the key feature that they have Killing spinors that are proportional to those preserved by a probe M5-brane wrapping a supersymmetric cycle in a special holonomy manifold (or equivalently by certain configurations of intersecting M5-branes). In this section we will define this more precisely.

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Our ansatz for the wrapped M5-brane geometries starts with the general ansatz for the metric given by

$$\mathrm{d}s^2 = L^{-1}\,\mathrm{d}s^2(\mathbb{R}^{1,d}) + \mathrm{d}s^2(\mathcal{M}_{10-d}), \tag{2.1}$$

where the warp factor L is a function of the coordinates on $\mathcal{M}_{10-d}$. In terms of the wrapped probe-M5-brane picture, the $\mathbb{R}^{1,d}$ factor corresponds to the unwrapped world-volume of the M5-brane and $\mathcal{M}_{10-d}$ corresponds to the wrapped M5-brane and the original geometry after the back-reaction has switched on. Since M5-branes are sources of magnetic flux, we also assume that the four-form flux F lies solely in $\mathcal{M}_{10-d}$, so that no electric flux, corresponding to membranes, is present.[2]

The ansatz for the Killing spinors is best described by considering an example. Let us take the case of a probe M5-brane wrapping a co-associative cycle in $\mathbb{R}^{1,3} \times \mathcal{M}_{G_2}$, where $\mathcal{M}_{G_2}$ is a seven-dimensional G_2-holonomy manifold. In this case the unwrapped worldvolume is two-dimensional, so the metric ansatz (2.1) has $d = 1$. The two directions in $\mathbb{R}^{1,3}$ orthogonal to the world volume we refer to as "overall transverse" directions. Let us introduce a frame

$$\mathrm{d}s^2 = 2e^+e^- + (e^1)^2 + \cdots + (e^9)^2, \tag{2.2}$$

where e^+ and e^- span the $\mathbb{R}^{1,1}$ unwrapped worldvolume directions and e^8 and e^9 are the overall transverse directions. Now consider the set of Killing spinors ϵ^i. In the case of the probe brane, the spacetime is $\mathbb{R}^{1,3} \times \mathcal{M}_{G_2}$ and admits four Killing spinors. The remaining seven basis one-forms can be chosen such that these Killing spinors satisfy the eleven-dimensional gamma-matrix projections

$$\Gamma^{1234}\epsilon^i = \Gamma^{3456}\epsilon^i = \Gamma^{1357}\epsilon^i = -\epsilon^i. \tag{2.3}$$

In this basis, the associative three-form Φ and co-associative four-form Υ on $\mathcal{M}_{G_2}$, take the standard form (A.3). If we now include the probe brane, the preserved supersymmetries will be eigenspinors of the chirality operator on the brane worldvolume. We can choose the orientation of the brane such that this condition reads

$$\Gamma^{+-1234}\epsilon^i = -\epsilon^i, \tag{2.4}$$

which is equivalent to $\Gamma^{+-}\epsilon^i = \epsilon^i$. This reduces the number of Killing spinors to two. These can be distinguished by, for instance, their eigenvalues under Γ^9

$$\Gamma^9\epsilon^1 = \epsilon^1, \qquad \Gamma^9\epsilon^2 = -\epsilon^2. \tag{2.5}$$

Together the spinors define a G_2 structure on the nine-dimensional manifold $\mathcal{M}_9 = \mathcal{M}_{G_2} \times \mathbb{R}^2$.

[2]For the cases of wrapped M5-branes that we will consider in this paper the electric flux is indeed vanishing for the explicitly known solutions of [8 – 11]. However, there are additional solutions in [10, 11], specifically for M5-branes wrapping 5-cycles or 4-cycles other than those considered here, where the M5-branes source electric flux and this would have to be properly taken into account in extending the analysis of this paper further.

wrapped brane	manifold	world-volume	supersymmetry
co-associative	G_2 holonomy	$\mathbb{R}^{1,1}$	$\mathcal{N} = (2,0)$
Kähler 4-cycle	CY_3	$\mathbb{R}^{1,1}$	$\mathcal{N} = (4,0)$
associative	G_2 holonomy	$\mathbb{R}^{1,2}$	$\mathcal{N} = 1$
SLAG	CY_3	$\mathbb{R}^{1,2}$	$\mathcal{N} = 2$
Kähler 2-cycle	CY_3	$\mathbb{R}^{1,3}$	$\mathcal{N} = 1$
Kähler 2-cycle	CY_2	$\mathbb{R}^{1,3}$	$\mathcal{N} = 2$

Table 1: Wrapped M5-brane geometries and their supersymmetry

The wrapped-brane spinor ansatz for the co-associative case is then that we consider those spacetimes which admit a pair of Killing spinors satisfying precisely the same projections as the probe brane geometry, namely (2.3), (2.4) and (2.5). It is important to note that this is not the most general ansatz for supersymmetric backgrounds with a warped $\mathbb{R}^{1,1}$ factor and two Killing spinors (which are chiral in $\mathbb{R}^{1,1}$, that is with $\mathcal{N} = (2,0)$ supersymmetry). The calibration projections (2.3) and (2.4) could be relaxed,[3] or, even if these hold, there is no reason, a priori, that both ϵ^i are eigenspinors of Γ^9.

In the co-associative example, note that one could choose the Killing spinors such that both were simultaneous eigenspinors of the five projection operators

$$\{\Gamma^{1234}, \Gamma^{3456}, \Gamma^{5678}, \Gamma^{1357}, \Gamma^{+-}\}\,. \tag{2.6}$$

This is actually characteristic of all the wrapped-brane Killing spinors we consider here and is one way of defining the class. The cases that we shall consider in this paper are summarised in table 1. The specific projections and conventions for the various cases are given in appendix A.

At this point, we will summarise our ansatz for the class of wrapped M5 brane spacetimes we consider. We demand that the metric contains a warped $\mathbb{R}^{1,d}$ factor, and is of the form (2.1). We demand that the flux has no electric components, and so lies entirely in $\mathcal{M}_{10-d}$. Finally, we demand that in each case the Killing spinors satisfy the appropriate probe brane projections, and in particular, that they are simultaneous eigenspinors of the five projection operators (2.6). It is worth emphasising that because we are imposing these projections, the G-structures that we use to characterise the wrapped brane solutions below are in fact globally defined.

In what follows in the next two sections, the main point is to note that there is a hierarchy of structures: structures with more supersymmetry can be viewed in a simple way as pairs of structures with less supersymmetry. In section 3, this will be used to derive the conditions for supersymmetry for all these cases in a simple way, starting only from the conditions for wrapping co-associative and associative cycles. In section 4, we will discuss

[3] For this particular case, but not the others we will consider, one particular way they can be relaxed corresponds to another wrapped M5-brane geometry, namely an M5-brane wrapping a Kähler four-cycle in a Calabi-Yau four-fold. This case also has $(2,0)$ supersymmetry in $\mathbb{R}^{1,1}$ but is distinguished from the co-associative case by the fact, for example, that both Killing spinors have the same eigenvalue under Γ^9. We do not treat this case in this paper because it is a case that should include electric flux.

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how the conditions can be understood in terms of generalised calibrations. In section 5 we will start discussing how we derive the AdS conditions.

3. Supersymmetry conditions for wrapped M5-brane spacetimes

In this section we will derive necessary and sufficient conditions on G-structures for all the wrapped-brane geometries given in table 1. In fact, many of these conditions have been written down before (although in some cases using stronger assumptions than ours). We will provide a unified treatment, emphasizing the feature that the structures with more supersymmetry can be viewed as pairs of structures with less supersymmetry. Indeed we will see that they all can be obtained from the associative and co-associative cases. This is analogous to the fact that the conditions required on G-structures for special holonomy manifolds in dimensions less than or equal to eight can be obtained from multiple $Spin(7)$ structures. In the following section, we will also show how they can be simply understood in terms of generalised calibrations. We will organise the discussion by the dimension $d+1$ of the unwrapped brane worldvolume.

3.1 Co-associative cycles in G_2 holonomy ($\mathbb{R}^{1,1}$, $\mathcal{N}=(2,0)$)

The general analysis of the conditions for supersymmetry for a pair or spinors satisfying the projections (2.3) and (2.4) was given in [17]. Using these conditions one can show that, assuming in addition only a warped $\mathbb{R}^{1,1}$ factor, supersymmetry then implies that the Killing spinors are eigenspinors of Γ^{5678} and F lies only in $\mathcal{M}_9$ (i.e. there is no electric flux). In other words, these parts of our ansatz need not be independently imposed for this case. The Killing spinors define a preferred $(G_2 \ltimes \mathbb{R}^7) \times \mathbb{R}^2$ structure in eleven dimensions [18]. That is to say, at generic points this is the stabilizer group of the two spinors.

The results of [17] imply that the metric on $\mathcal{M}_9$ is compatible with an integrable product structure (though the manifold is not necessarily a product)

$$ds^2(\mathcal{M}_9) = ds^2(\mathcal{M}_{G_2}) + L^2(dy_1^2 + dy_2^2), \tag{3.1}$$

where both L and Φ depend on all coordinates of $\mathcal{M}_9$ and there is a G_2 structure Φ on $\mathcal{M}_{G_2}$. If we define the orientation

$$\epsilon = \tfrac{1}{7}\Phi \wedge \Upsilon \wedge \mathrm{vol}_Y \tag{3.2}$$

where $\mathrm{vol}_Y = L^2(dy_1 \wedge dy_2)$, the remaining conditions for supersymmetry may then be rewritten as

$$\begin{aligned} \mathrm{vol}_Y \wedge d\Phi &= 0, & (3.3)\\ d(L^{-1}\Phi \wedge \Upsilon) &= 0, & (3.4)\\ \Phi \wedge d\Phi &= 0, & (3.5)\\ \star_9 F &= L d(L^{-1}\Upsilon). & (3.6) \end{aligned}$$

We see that the metric (3.1) is conformally flat on the overall transverse directions $\mathrm{d}y_1$ and $\mathrm{d}y_2$ (corresponding to e^8 and e^9 in the conventions of appendix A). Note that these conditions imply the following useful expression for the flux

$$(\star_2 v) \wedge \mathrm{d}\Phi = v \wedge F \tag{3.7}$$

for any one-form v lying in the overall transverse space and where $\star_2$ is the Hodge star on the transverse space defined using the orientation vol_Y. As we will see in the next section, the supersymmetry conditions (3.3)-(3.6) can be understood in terms of generalised calibrations. This perspective also provides a simple way to obtain (3.7).

3.2 Kähler-4 cycles in SU(3) holonomy ($\mathbb{R}^{1,1}$, $\mathcal{N} = (4,0)$)

For a probe M5-brane wrapping a Kähler four-cycle in an SU(3) holonomy manifold, the Killing spinors define a preferred $(SU(3) \ltimes \mathbb{R}^6) \times \mathbb{R}^3$ structure in eleven dimensions [18]. From appendix A, we see that the Killing spinor projections (A.6) and (A.9) are equivalent to a pair of co-associative projections with the G_2 structures given by (A.8). Thus the corresponding supersymmetry conditions can be derived from the conditions (3.3)–(3.6) for the pair of G_2 structures $\Phi_\pm$.

After some manipulations one finds that the additional overall transverse direction is given by $e^7 = L\mathrm{d}y_3$ so that the metric takes the product form

$$\mathrm{d}s^2(\mathcal{M}_9) = \mathrm{d}s^2(\mathcal{M}_{\mathrm{SU}(3)}) + L^2(\mathrm{d}y_1^2 + \mathrm{d}y_2^2 + \mathrm{d}y_3^2), \tag{3.8}$$

where there is an SU(3) structure on $\mathcal{M}_{\mathrm{SU}(3)}$ and L depends on all coordinates of $\mathcal{M}_9$. Fixing the orientation

$$\epsilon = \tfrac{1}{6} J \wedge J \wedge J \wedge \mathrm{vol}_Y, \tag{3.9}$$

with $\mathrm{vol}_Y = L^3 \mathrm{d}y_1 \wedge \mathrm{d}y_2 \wedge \mathrm{d}y_3$, the remaining supersymmetry conditions read

$$\mathrm{vol}_Y \wedge \mathrm{d}(LJ) = 0, \tag{3.10}$$

$$\mathrm{d}\Omega = 0, \tag{3.11}$$

$$\star_9 F = L\mathrm{d}\left(L^{-1}\tfrac{1}{2} J \wedge J\right). \tag{3.12}$$

These are in agreement with the results of [19, 20] which were obtained with some additional assumptions about the form of the backgrounds. It is worth noting that the conditions imply that we have a Kähler metric on $\mathcal{M}_{\mathrm{SU}(3)}$. The complex structure is independent of y_i, while the Kähler form LJ is a function of the y_i. Again one can also derive a useful expression for the flux

$$(\star_3 v) \wedge L^{-1}\mathrm{d}(LJ) = -v \wedge F \tag{3.13}$$

for any one-form v lying in the overall transverse space spanned by $\mathrm{d}y_i$, and where $\star_3$ is defined using vol_Y.

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3.3 Associative cycles in G_2 holonomy ($\mathbb{R}^{1,2}$, $\mathcal{N}=1$)

The general local analysis of the minimal supersymmetry conditions with a warped $\mathbb{R}^{1,2}$ factor are given in [2] (a discussion of some global issues can be found in [3]). The Killing spinors can always be chosen to satisfy the G_2 structure conditions (A.2). As shown in [2], requiring F to lie on $\mathcal{M}_8$, i.e. no electric flux, then implies in addition the associative calibration projection (A.5) and hence that we have a wrapped-brane geometry. The Killing spinors define a preferred G_2 structure in eleven dimensions.

Summarising the conditions of [2] for supersymmetry in this case, the metric is again a product

$$\mathrm{d}s^2(\mathcal{M}_8) = \mathrm{d}s^2(\mathcal{M}_{G_2}) + L^2\mathrm{d}y^2 \tag{3.14}$$

where L depends on all coordinates on $\mathcal{M}_8$. Fixing the orientation

$$\epsilon = \tfrac{1}{7}\Phi\wedge\Upsilon\wedge v, \tag{3.15}$$

where $v = L\mathrm{d}y$ is the overall transverse direction, we have

$$v\wedge\mathrm{d}(L^{-1}\Upsilon) = 0, \tag{3.16}$$

$$\mathrm{d}(L^{-5/2}\Phi\wedge\Upsilon) = 0, \tag{3.17}$$

$$\Phi\wedge\mathrm{d}\Phi = 0, \tag{3.18}$$

$$\star_8 F = -L^{3/2}\mathrm{d}(L^{-3/2}\Phi). \tag{3.19}$$

These imply the expression for the flux

$$L\mathrm{d}(L^{-1}\Upsilon) = -v\wedge F. \tag{3.20}$$

3.4 SLAG cycles in SU(3) holonomy ($\mathbb{R}^{1,2}$, $\mathcal{N}=2$)

The supersymmetry conditions for this case are given in [2]. Here we observe that they can be also obtained from our co-associative conditions.

The spinor projections for a probe brane wrapping a SLAG cycle in an SU(3)-holonomy manifold (A.6) and (A.11) define a preferred SU(3) structure in eleven dimensions. They are equivalent to a pair of co-associative projections. From (A.11), one sees the two corresponding co-associative structures are given by G_2-structures $\Phi_\pm$ (A.8), together with exchanging e^+ and e^-.

Since the one-form e^7 lies along the unwrapped worldvolume, we demand that $e^7 = L^{-1/2}\mathrm{d}x_2$ and in addition that the flux has no component in this direction. Then demanding that the two G_2 structures satisfy the supersymmetry conditions (3.3)–(3.6), we recover the results of [2]. In particular, we find that the eight-dimensional metric is a product

$$\mathrm{d}s^2(\mathcal{M}_8) = \mathrm{d}s^2(\mathcal{M}_{\mathrm{SU}(3)}) + L^2(\mathrm{d}y_1^2 + \mathrm{d}y_2^2) \tag{3.21}$$

where L depends on all coordinates on $\mathcal{M}_8$. Defining the orientation

$$\epsilon = \tfrac{1}{6}J\wedge J\wedge J\wedge\mathrm{vol}_Y, \tag{3.22}$$

where $\mathrm{vol}_Y = L^2 \mathrm{d}y_1 \wedge \mathrm{d}y_2$, the remaining supersymmetry conditions read

$$\mathrm{vol}_Y \wedge \mathrm{d}\,\mathrm{Im}\,\Omega = 0, \tag{3.23}$$

$$\mathrm{d}(L^{-1/2}J) = 0, \tag{3.24}$$

$$\mathrm{d}\,\mathrm{Re}\,\Omega \wedge \mathrm{Re}\,\Omega = 0, \tag{3.25}$$

$$\star_8 F = L^{3/2}\mathrm{d}(L^{-3/2}\,\mathrm{Re}\,\Omega). \tag{3.26}$$

These imply the expression for the flux

$$(\star_2 v) \wedge \mathrm{d}\,\mathrm{Im}\,\Omega = v \wedge F, \tag{3.27}$$

for any v lying in the overall transverse space.

3.5 Kähler-2 cycles in SU(3) holonomy ($\mathbb{R}^{1,3}$, $\mathcal{N} = 1$)

For the case of an M5-brane wrapping a Kähler 2-cycle in an SU(3)-holonomy manifold, the Killing spinors define a preferred SU(3) structure in eleven dimensions. As in the SLAG case, we may derive the supersymmetry conditions directly from the conditions corresponding to wrapping associative cycles. The two-cycle spinor projections (A.6) and (A.10) are equivalent to a pair of associative projections with G_2 structures given by (A.8). Since e^7 lies along the unwrapped worldvolume, we demand that $e^7 = L^{-1/2}\mathrm{d}x_3$ and we also demand that the flux has no component along e^7.

One finds the product metric

$$\mathrm{d}s^2(\mathcal{M}_7) = \mathrm{d}s^2(\mathcal{M}_{\mathrm{SU}(3)}) + L^2\mathrm{d}y^2 \tag{3.28}$$

where L depends on all coordinates on $\mathcal{M}_7$. Defining the orientation

$$\epsilon = \tfrac{1}{6} J \wedge J \wedge J \wedge v, \tag{3.29}$$

where $v = L\mathrm{d}y$, the remaining supersymmetry conditions read

$$v \wedge \mathrm{d}(L^{-1}J \wedge J) = 0, \tag{3.30}$$

$$\mathrm{d}(L^{-3/2}\Omega) = 0, \tag{3.31}$$

$$\star_7 F = -L^2\mathrm{d}(L^{-2}J). \tag{3.32}$$

These are consistent with the results of [21], where additional assumptions about the form of the background were made. Note that these conditions imply that $\mathcal{M}_{\mathrm{SU}(3)}$ is a complex manifold, with a complex structure independent of y_i and an hermitian metric dependent on y_i. The conditions also imply

$$\tfrac{1}{2}L\mathrm{d}\left(L^{-1}J \wedge J\right) = -v \wedge F. \tag{3.33}$$

3.6 Kähler-2 cycles in SU(2) holonomy ($\mathbb{R}^{1,3}$, $\mathcal{N} = 2$)

The final case we consider is that of an M5-brane wrapping a Kähler two-cycle in an SU(2)-holonomy manifold. The Killing spinors for this case define a preferred SU(2) structure in eleven dimensions. The spinor projections (A.12) and (A.15) are equivalent to a pair of Kähler two-cycle in SU(3) holonomy projections, with the two corresponding SU(3) structures given by (A.14).

Using the conditions for supersymmetry for each of the SU(3) wrapped brane geometries that were derived in the last subsection, one can show that $e^5 = L\mathrm{d}y_2$ and $e^6 = L\mathrm{d}y_3$ so that the seven-dimensional metric is a product

$$\mathrm{d}s^2(\mathcal{M}_7) = \mathrm{d}s^2(\mathcal{M}_{\mathrm{SU}(2)}) + L^2(\mathrm{d}y_1^2 + \mathrm{d}y_2^2 + \mathrm{d}y_3^2) \tag{3.34}$$

where L depends on all coordinates on $\mathcal{M}_7$. Defining the orientation

$$\epsilon = \tfrac{1}{2}J^1 \wedge J^1 \wedge \mathrm{vol}_Y, \tag{3.35}$$

where $\mathrm{vol}_Y = L^3\mathrm{d}y_1 \wedge \mathrm{d}y_2 \wedge \mathrm{d}y_3$, the remaining supersymmetry conditions read

$$\mathrm{d}(L^{-1/2}J^2) = \mathrm{d}(L^{-1/2}J^3) = 0, \tag{3.36}$$

$$\mathrm{vol}_Y \wedge \mathrm{d}(LJ^1) = 0, \tag{3.37}$$

$$\star_7 F = L^2\mathrm{d}(L^{-2}J^1). \tag{3.38}$$

These conditions were first derived in [22] (extending the results of [23]). The combination $J^3 + \mathrm{i}J^2$ defines a complex structure on $\mathcal{M}_{\mathrm{SU}(2)}$ independent of y_i, while LJ^1 defines a Kähler metric at fixed y_i. The conditions also imply

$$(\star_3 v) \wedge L^{-1}\mathrm{d}(LJ^1) = -v \wedge F, \tag{3.39}$$

where v is any one-form in the overall transverse space.

4. Relation to generalised calibrations

A key early paper highlighting the role of generalised calibrations [24] in describing classes of supersymmetric supergravity geometries is the work by Cho et al. [19]. This was subsequently explored in the context of IIB supergravity in [25]. In [16] it was shown that generic eleven-dimensional supersymmetric solutions admit generalised calibrations. The fact that, for certain cases of solutions, *all* of the conditions for supersymmetry can be understood in terms of generalised calibrations was discussed in [26]. For some of the cases that we consider in this paper, the observation that a subset of the supersymmetry conditions are related to generalised calibrations was made in [20, 22, 27]. The relation to generalised calibrations of warped supersymmetric geometries of the form $\mathbb{R}^{1,2} \times \mathcal{M}_8$ was discussed in detail in [2]. In this section we will briefly show that in fact all of the supersymmetry conditions for the wrapped brane geometries can be interpreted this way.

A calibrating p-form Ξ on a Riemannian manifold $\mathcal{M}$ has the property that the metric-induced volume form on any oriented p-dimensional subspace ξ of $T_x\mathcal{M}$ is greater than or equal to the restriction of Ξ to ξ. A p-dimensional submanifold C_p is calibrated if the bound is saturated, $\Xi|_{T_x C_p} = \mathrm{vol}_{T_x C_p}$, everywhere on C_p. Conventionally the calibrating form is also required to be closed. This then implies that the calibrated cycle has minimum volume in its homology class. Crucially all the structure forms defining special holonomy manifolds are calibrating forms. The different supersymmetric cycles (associative, co-associative, Kähler, SLAG) we have been discussing are all calibrated cycles.

For a generalised calibration [24], one retains the algebraic condition relating Ξ to the volume, but generalises the differential condition so that Ξ is no longer closed. Instead $\mathrm{d}\Xi$ involves the flux, and any warping factor if the spacetime is a warped product $\mathbb{R}^{1,d} \times \mathcal{M}$ as in (2.1). The point is that the calibrated cycles now extremize not their volume but rather the brane energy, including for instance the contribution from the flux, for probe branes wrapping the corresponding cycle. In the case of M5-branes, one gets conditions like

$$L^m \mathrm{d}\left(L^{-m}\Xi\right) = \star_{10-d} F, \tag{4.1}$$

for some $m \in \mathbb{Q}$. In a supersymmetric background, the generalised calibrated cycles are supersymmetric.

Note that for each of the wrapped brane geometries, the supersymmetry conditions contain one condition of the form (4.1). In fact, the remaining supersymmetry conditions can also be interpreted as generalised calibrations related to other ways in which probe M5-branes (or M2-branes) can wrap various calibrated cycles whilst preserving supersymmetry. (In the context of type II supergravity this is discussed in more detail in the introduction and conclusion of [26]). To see this we will show, equivalently, that the supersymmetry conditions can be obtained from the generalised calibration conditions arising from minimally supersymmetric solutions of $D = 11$ supergravity [16].

The essential point is that the assumption that each Killing spinor is an eigenspinor of the five projection operators (2.6) implies that each spinor defines a local $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ structure, with the structures fitting together in a very simple way. Explicitly, following [16], if ϵ has eigenvalue $+1$ under Γ^{+-} and -1 under all the other projectors, the corresponding structure can be written as

$$\begin{aligned} K &= e^+, \\ \Omega &= K \wedge v, \\ \Sigma &= K \wedge \phi, \end{aligned} \tag{4.2}$$

where $v = e^9$ and ϕ is the $Spin(7)$ invariant

$$\begin{aligned} \phi = &- e^{1234} - e^{1256} - e^{1278} - e^{3456} - e^{3478} - e^{5678} \\ &- e^{1357} + e^{1368} + e^{1458} + e^{1467} + e^{2358} + e^{2367} + e^{2457} - e^{2468}. \end{aligned} \tag{4.3}$$

One can then show [16], that the Killing spinor equation for ϵ implies set of differential conditions on (K, Ω, Σ),

$$\begin{aligned} \mathrm{d}K &= \tfrac{2}{3} i_\Omega F + \tfrac{1}{3} i_\Sigma \star F, \\ \mathrm{d}\Omega &= i_K F, \\ \mathrm{d}\Sigma &= i_K \star F - \Omega \wedge F. \end{aligned} \tag{4.4}$$

One can view these as a set of generalised calibration conditions. Since K is null, the first one is associated to massless particles, the second is associated to wrapped M2-branes (coupling to electric flux) and the third to wrapped M5-branes (coupling to magnetic flux).

Now consider, for instance, the case of a co-associative calibration. We now have a pair of Killing spinors. Following our discussion in section 2, each has positive eigenvalue under Γ^{+-} and negative eigenvalue under Γ^{1234}, Γ^{3456} and Γ^{1357}. They are distinguished by their eigenvalue under Γ^{5678} or equivalently Γ^9. From this perspective, each spinor defines a different $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ local structure. Explicitly, these are given by (4.2) with $\{K = e^+, \phi = \phi_\pm, v = \pm e^9\}$ where

$$\phi_\pm = \mp\Phi \wedge e^8 - \Upsilon, \tag{4.5}$$

with Φ the three-form defining the G_2 structure. In fact, more generally, by taking constant linear combinations of ϵ^1 and ϵ^2 we get a family of $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ structures $\{K = e^+, \phi = \phi(\theta), v = v(\theta)\}$ where

$$\begin{aligned} v(\theta) &= \cos\theta e^9 - \sin\theta e^8, \\ \phi(\theta) &= -\Phi \wedge (\cos\theta e^8 + \sin\theta e^9) - \Upsilon. \end{aligned} \tag{4.6}$$

and θ is constant.

Supersymmetry implies that the generalised calibration conditions (4.4) must be satisfied for all these structures. This then gives us a simple way to derive, in this case, the wrapped-brane geometry supersymmetry conditions for a co-associative calibration. Explicitly we start with the metric ansatz (2.1). Writing $K = e^+ = L^{-1}\mathrm{d}x^+$ and given that the flux lies solely on $\mathcal{M}_9$, the M2-brane calibration condition for Ω with general θ implies that

$$\mathrm{d}(L^{-1}v(\theta)) = 0 \tag{4.7}$$

or locally $e^8 = L\mathrm{d}y_1$ and $e^9 = L\mathrm{d}y_2$. Given the orientation (3.2), the M5-brane Σ calibration condition gives

$$\begin{aligned} L\mathrm{d}(L^{-1}\Upsilon) &= \star_9 F, \\ L\mathrm{d}(L^{-1}v(\theta) \wedge \Phi) &= - \star_2 v(\theta) \wedge F, \end{aligned} \tag{4.8}$$

for all θ. After some manipulations one can show that these imply the set of supersymmetry conditions (3.3)–(3.6) given in the previous section. Note that in this case the K calibration condition is implied by the M2-brane and M5-brane calibration conditions.

Thus we see that the calibration conditions (4.4) for each spinor are in fact necessary and sufficient for supersymmetry of the wrapped M5-brane geometry. A similar calculation can be used to derive the supersymmetry conditions for the wrapped-brane geometries related to associative calibrations. We saw in the previous section that all the other wrapped-brane geometry supersymmetry conditions could be derived from this basic pair, and hence, ultimately from the calibration conditions (4.4) (in fact limited only to the Ω and Σ calibrations). From this perspective, the conditions for supersymmetry are equivalent to requiring that all the possible structure forms, compatible with the G-structure of the background, are actually generalised calibrations.

Physically one can view the set of calibration conditions as corresponding to all the possible additional supersymmetric wrapped probe M5-branes and probe M2-branes compatible with the supersymmetry of the wrapped-brane geometry. In the example above the first conditions (4.7) correspond to calibration conditions for M2-branes spanning e^+, e^- and $v(\theta)$. The second set of conditions (4.8), correspond to M5-branes spanning e^+, e^- and a co-associative cycle in $\mathcal{M}$, or e^+, e^-, $v(\theta)$ and an associative cycle in $\mathcal{M}$.

5. AdS spacetimes from wrapped-brane spacetimes

In this section, we will discuss how to obtain AdS backgrounds from the wrapped-brane geometries discussed thus far. Once we have formulated the AdS limit, we may simply insert it in the wrapped-brane supersymmetry conditions to obtain the conditions for supersymmetry of the AdS spacetimes.

In Poincaré coordinates a general AdS_{d+2} spacetime can be written as

$$\begin{aligned} \mathrm{d}s^2 &= \lambda^{-1}\mathrm{d}s^2(AdS_{d+2}) + \mathrm{d}s^2(\mathcal{N}_{9-d}) \\ &= \lambda^{-1}\left[\mathrm{e}^{-2mr}\mathrm{d}s^2(\mathbb{R}^{1,d}) + \mathrm{d}r^2\right] + \mathrm{d}s^2(\mathcal{N}_{9-d}), \end{aligned} \tag{5.1}$$

This can be obtained from the wrapped geometries (2.1) by demanding

$$\begin{aligned} L &= \mathrm{e}^{2mr}\lambda, \\ \mathrm{d}s^2(\mathcal{M}_{10-d}) &= \lambda^{-1}\mathrm{d}r^2 + \mathrm{d}s^2(\mathcal{N}_{9-d}), \end{aligned} \tag{5.2}$$

and where the warp factor λ is taken to be a function of the coordinates on $\mathrm{d}s^2(\mathcal{N}_{9-d})$. Note that the vector $\partial/\partial r$ is both Killing and hypersurface orthogonal on $\mathrm{d}s^2(\mathcal{M}_{10-d})$. We also assume that the flux F lies solely in $\mathcal{N}_{9-d}$, and is independent of the AdS radial coordinate, so that the full solution preserves the AdS isometries.

To analyse this reduction of $\mathcal{M}_{10-d}$ to $\mathcal{N}_{9-d}$, we note that in all the wrapped-brane geometries the metric took the particular product form where the overall transverse directions are conformally flat,

$$\mathrm{d}s^2(\mathcal{M}_{10-d}) = \mathrm{d}s^2(\mathcal{M}_G) + L^2\left(\mathrm{d}t^2 + t^2\mathrm{d}\Omega^2_{q-1}\right), \tag{5.3}$$

where we have introduced polar coordinates on the q-dimensional transverse space, so $\mathrm{d}\Omega^2_{q-1}$ is the round metric on the unit sphere S^{q-1}. For the cases of interest $q = 1, 2, 3$. Note in addition, that there is G-structure on $\mathcal{M}_G$ in each case. This product structure means that generically the radial vector $\partial/\partial r$ will split into a part in $\mathcal{M}_G$ and a part in the overall transverse space. In particular, we can write the AdS unit radial one-form as

$$\lambda^{-1/2}\mathrm{d}r = \sin\theta\,\hat{u} + \cos\theta\,\hat{v}, \tag{5.4}$$

where $\hat{u}$ is a unit one-form in $\mathcal{M}_G$, and $\hat{v}$ is a unit one-form in the overall transverse space.

We will make the assumption that $\hat{v}$ is given by

$$\hat{v} = L\mathrm{d}t, \tag{5.5}$$

and so lies along the radial direction t of the conformally flat overall transverse space. In addition we will assume that the rotation angle θ must be independent of the AdS radial coordinate. As we will see below, these assumptions lead to geometries with at least part[4] of the R-symmetry of the field theory realised as isometries of the sphere S^{q-1}, as one might expect. Here we have presented (5.5) and the r-independence of θ as assumptions, though we emphasise that rather stronger statements regarding the generality, or otherwise, of our AdS limit can be made. As we discuss in more detail in appendix B, for the case of one overall transverse direction, the rotation angle θ is in fact necessarily independent of r, so in this case our AdS limit is in fact the most general way of obtaining an AdS geometry from the wrapped brane spacetime. For the case of two or three overall transverse directions our results are slightly weaker, but we show that with a suitable assumption of r-independence of the frame rotation, the part of the AdS radial direction which lies in the overall transverse space must in fact lie entirely along the radial direction of the overall transverse space, as in (5.5).

Now, introducing the orthogonal combination

$$\hat{\rho} = \cos\theta\, \hat{u} - \sin\theta\, \hat{v}, \tag{5.6}$$

the fact that $\mathrm{d}t$ is closed, and θ is independent of r, then implies that

$$\hat{\rho} = \frac{\lambda}{2m\sin\theta}\mathrm{d}(\lambda^{-3/2}\cos\theta). \tag{5.7}$$

Defining a new coordinate $\rho = \lambda^{-3/2}\cos\theta$ one then has the relation $t = -(\rho/2m)\mathrm{e}^{-2mr}$ and

$$\begin{aligned}\hat{\rho} &= \frac{\lambda \mathrm{d}\rho}{2m\sqrt{1-\lambda^3\rho^2}},\\ \hat{u} &= \lambda^{-1/2}\sqrt{1-\lambda^3\rho^2}dr + \frac{\lambda^{5/2}\rho d\rho}{2m\sqrt{1-\lambda^3\rho^2}}.\end{aligned} \tag{5.8}$$

Extracting the AdS factor, one finds that the metric $\mathrm{d}s^2(\mathcal{N}_{9-d})$ then takes the form

$$\mathrm{d}s^2(\mathcal{N}_{9-d}) = \mathrm{d}s^2(\mathcal{M}_{G'}) + \frac{\lambda^2}{4m^2}\left(\frac{\mathrm{d}\rho^2}{1-\lambda^3\rho^2} + \rho^2\mathrm{d}\Omega^2_{q-1}\right), \tag{5.9}$$

where $\mathrm{d}s^2(\mathcal{M}_{G'})$ is defined via

$$\mathrm{d}s^2(\mathcal{M}_G) = \mathrm{d}s^2(\mathcal{M}_{G'}) + \hat{u}\otimes\hat{u}. \tag{5.10}$$

The G'-structure on $\mathcal{M}_{G'}$ is a reduction of the G-structure on M_G, defined by picking out the particular one-form $\hat{u}$. It is useful in what follows to define a normalised volume form on the transverse sphere S^{q-1} in (5.2)

$$\widehat{\mathrm{vol}}(S^{q-1}) = \left(\frac{\lambda\rho}{2m}\right)^{q-1}\mathrm{vol}(S^{q-1}), \tag{5.11}$$

[4] For the cases we consider of M5-branes wrapping Kähler cycles, the complex structure on $\mathcal{M}_G$ acting on $\hat{u}$ picks out another direction in $\mathcal{M}_G$ and this also contributes to the R-symmetry.

where $\mathrm{vol}(S^{q-1})$ is the volume form on the unit sphere.

Given the supersymmetry conditions on the original space $\mathcal{M}_{10-d}$ it is then straightforward to take (5.2) with $\mathrm{d}s^2(\mathcal{N}_{9-d})$ given by (5.9), demand that the flux has no components along the AdS radial direction, and hence derive the supersymmetry conditions for an AdS_{d+2} geometry in terms of the G'-structure. It is worth noting that this G'-structure is, in general, only locally defined, since there can be points where $\sin\theta = 0$ and hence the vector $\hat{u}$ is ill-defined.

The discussion thus far has been for the generic case where $\partial/\partial r$ lies partly in $\mathcal{M}_G$ and partly in the overall transverse space. There are two special cases one should also consider. First $\partial/\partial r$ could lie entirely in $\mathcal{M}_G$. This is excluded since it is inconsistent with $\partial/\partial r$ being Killing, since from (5.2) and (5.3) we see that the overall transverse space would then have an explicit dependence on r.

The second possibility is that $\partial/\partial r$ lies entirely in the overall transverse space. Note that in all the cases analysed in section 3 we have a condition of the form

$$\mathrm{d}(L^m \,\mathrm{vol}_G) = 0, \tag{5.12}$$

for some $m \in \mathbb{Q}$, where vol_G is the volume form on $\mathcal{M}_G$. Since $\partial/\partial r$ is Killing and assuming it lies solely in the overall transverse space, we have that vol_G is independent of r. This implies, provided $m \neq 0$, that

$$\mathrm{d}r \wedge \mathrm{vol}_G = 0, \tag{5.13}$$

which is impossible. Note that $m = 0$ only in the case of Kähler-4 calibrations in SU(3) holonomy. Thus only in this one special case do we need to consider the case where $\partial/\partial r$ lies solely in the overall transverse space. This is discussed separately in section 6.2.

Let us end this section by noting that for all the AdS geometries we obtain from the wrapped-brane spacetimes, supersymmetry implies that all equations of motion and the Bianchi identity are identically satisfied. From section 3 it is clear that for all wrapped-brane geometries we have

$$\star_{10-d}F = L^{r_1}\mathrm{d}\left(L^{-r_1}\Xi_1\right), \tag{5.14}$$

$$v \wedge F = (\star_p v) \wedge L^{r_2}\mathrm{d}\left(L^{-r_2}\Xi_2\right), \tag{5.15}$$

for some $r_1, r_2 \in \mathbb{Q}$, some calibration forms Ξ_1 and Ξ_2, and any one-form v in the overall transverse space. By taking the exterior derivative of (5.14), one automatically satisfies the equation of motion for F for any wrapped-brane geometry. Generically, the Bianchi identity is not satisfied as a consequence of the wrapped brane supersymmetry conditions, and must be imposed. Imposing the Bianchi identity, the results of [15, 17], then imply that the Einstein equations are identically satisfied, with the possible exception of the $++$ component in the co-associative and Kähler-4 cases; however because we have assumed a warped Minkowski factor, it is easy to check that these components are in fact satisfied. Thus to guarantee a solution of the field equations for the wrapped brane spacetimes, we need only impose the Bianchi identity in addition to the supersymmetry conditions. By taking the AdS limit of (5.15), we may easily deduce the flux in each case, and in each case we have verified that the Bianchi identity for the flux is identically satisfied in the AdS limit. Therefore the supersymmetry conditions in the AdS limit are necessary and sufficient to guarantee a solution of all the field equations and the Bianchi identity.

In the following sections, we present the supersymmetry conditions for AdS_{d+2} spacetimes, using the reduction (5.2), for each of the different wrapped brane geometries. The derivations are straightforward but a bit long, so we just give some sample calculations for a representative example in appendix C.

6. Supersymmetric AdS_3 spacetimes

In this section, we will use the reduction discussed in the previous section to obtain the conditions for the supersymmetric AdS_3 spacetimes contained in the wrapped brane geometries with an $\mathbb{R}^{1,1}$ factor. Specifically, these will correspond to M5-branes wrapping co-associative cycles in G_2-holonomy manifolds and Kähler four-cycles in SU(3)-holonomy manifolds.

6.1 AdS_3 spacetimes from wrapping co-associative cycles

These geometries will be dual to two-dimensional SCFTs with a chiral $\mathcal{N} = (2,0)$ supersymmetry, and hence with a U(1) R-symmetry.

The overall transverse space is two-dimensional in this case, and so we have $q = 2$ in (5.3). Picking the unit one-form $\hat{u}$ in $\mathcal{M}_{G_2}$ breaks the local structure to $G' = \mathrm{SU}(3)$, defined by J and Ω, as given in appendix A with $e^7 = \hat{u}$. Thus we have

$$\mathrm{d}s^2(\mathcal{N}_8) = \mathrm{d}s^2(\mathcal{M}_{\mathrm{SU}(3)}) + \frac{\lambda^2}{4m^2}\left(\frac{\mathrm{d}\rho^2}{1-\lambda^3\rho^2} + \rho^2\mathrm{d}\phi^2\right), \tag{6.1}$$

where ϕ is a coordinate on the S^1. We define the orientation

$$\epsilon = \tfrac{1}{6} J \wedge J \wedge J \wedge \hat{\rho} \wedge \widehat{\mathrm{vol}}(S^1), \tag{6.2}$$

where we recall that the normalised volume $\widehat{\mathrm{vol}}(S^1)$ was defined in (5.11). The remaining independent conditions for supersymmetry turn out to be as follows:

$$\mathrm{d}\left(\frac{1}{\lambda^{3/2}\rho} J \wedge \hat{\rho} - \mathrm{Im}\,\Omega\right) = 0, \tag{6.3}$$

$$\mathrm{d}\left(\frac{1}{2\lambda} J \wedge J + \lambda^{1/2}\rho\,\mathrm{Re}\,\Omega \wedge \hat{\rho}\right) = 0. \tag{6.4}$$

From the flux condition (3.7), we find

$$F = -\frac{1}{\lambda\rho}\widehat{\mathrm{vol}}(S^1) \wedge \mathrm{d}\left(\lambda^{-1/2}\sqrt{1-\lambda^3\rho^2}J\right). \tag{6.5}$$

It follows from the supersymmetry conditions (6.3) and (6.4) that the rotational Killing vector on S^1 is a Killing vector of the full solution that preserves the flux. Therefore the supersymmetry conditions imply that generically the isometry group of the AdS limit, and hence the R-symmetry of the general dual SCFT, is U(1), as expected. In section 9 we will recover the explicit supersymmetric solution of [10] using these results.

6.2 AdS_3 spacetimes from wrapping Kähler four-cycles in SU(3) holonomy

For the case of wrapped brane geometries corresponding to Kähler four-cycles in spaces with SU(3) holonomy there are three overall transverse directions and so $q = 3$. As we discussed in the previous section, there are two distinct ways of taking the AdS_3 limit and we shall discuss both of them. The AdS_3 geometries will be dual to two-dimensional SCFTs with a chiral $\mathcal{N} = (4, 0)$ supersymmetry.

AdS radial direction from the overall transverse space Demanding that the AdS radial direction lies entirely in the overall transverse space implies that λ is constant and that $\partial/\partial r$ lies along the radial direction of the overall transverse space. Hence, rescaling so $\lambda = 1$, instead of (5.9), we have

$$\mathrm{d}s^2(\mathcal{N}_8) = \mathrm{d}s^2(\mathcal{M}_{\mathrm{SU}(3)}) + \frac{1}{4m^2}\mathrm{d}\Omega_2^2. \tag{6.6}$$

The supersymmetry conditions then imply that

$$\mathrm{d}J = \mathrm{d}\Omega = 0, \tag{6.7}$$

and $F = -\operatorname{vol}(S^2) \wedge J/2m$. Therefore $\mathcal{M}_{\mathrm{SU}(3)}$ is Calabi–Yau and, in addition, the rotational Killing vectors on the S^2 are Killing vectors of the full solution and also preserve the flux. This is the well known $AdS_3 \times S^2 \times CY_3$ solution. Generically, the isometry group of the space transverse to the AdS factor is SU(2).

Generic AdS radial direction Generically the AdS radial direction has a component in $\mathcal{M}_{\mathrm{SU}(3)}$ and a component in the overall transverse space as discussed in section 5. The component $\hat{u}$ in $\mathcal{M}_{\mathrm{SU}(3)}$ reduces the structure to $G' = \mathrm{SU}(2)$ in five dimensions, where in the conventions of appendix A we have $e^6 = \hat{u}$. Such a structure is defined by a triplet of two-forms J^i, defining a conventional four-dimensional SU(2) structure together with an additional one-form $\hat{w} = e^5$ (in the conventions of appendix A). Thus we have

$$\mathrm{d}s^2(\mathcal{N}_8) = \mathrm{d}s^2(\mathcal{M}_{\mathrm{SU}(2)}) + \hat{w} \otimes \hat{w} + \frac{\lambda^2}{4m^2}\left(\frac{\mathrm{d}\rho^2}{1 - \lambda^3\rho^2} + \rho^2\mathrm{d}\Omega_2^2\right), \tag{6.8}$$

where there is an SU(2) structure on the four-dimensional space $\mathcal{M}_{\mathrm{SU}(2)}$.

Defining the orientation

$$\epsilon = \tfrac{1}{6}J^i \wedge J^i \wedge \hat{w} \wedge \hat{\rho} \wedge \widehat{\mathrm{vol}}(S^2), \tag{6.9}$$

the conditions for supersymmetry are

$$\mathrm{d}\left(\lambda^{-1/2}\sqrt{1 - \lambda^3\rho^2}J^2\right) = 0, \tag{6.10}$$

$$\mathrm{d}\left(\lambda^{-1/2}\sqrt{1 - \lambda^3\rho^2}J^3\right) = 0, \tag{6.11}$$

$$\mathrm{d}\left(\lambda\rho J^1 + \lambda^{-1/2}\hat{w} \wedge \hat{\rho}\right) = 0, \tag{6.12}$$

$$J^3 \wedge \mathrm{d}\left(\frac{\lambda^{1/2}}{\sqrt{1 - \lambda^3\rho^2}}\hat{w}\right) = J^2 \wedge \mathrm{d}\left(\frac{\lambda^2\rho}{\sqrt{1 - \lambda^3\rho^2}}\hat{\rho}\right), \tag{6.13}$$

$$J^2 \wedge \mathrm{d}\left(\frac{\lambda^{1/2}}{\sqrt{1 - \lambda^3\rho^2}}\hat{w}\right) = -J^3 \wedge \mathrm{d}\left(\frac{\lambda^2\rho}{\sqrt{1 - \lambda^3\rho^2}}\hat{\rho}\right), \tag{6.14}$$

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and the flux is given by

$$F = -\frac{1}{\lambda^2\rho^2}\,\widehat{\mathrm{vol}}(S^2) \wedge \left[\mathrm{d}\left(\lambda^{1/2}\rho\sqrt{1-\lambda^3\rho^2}\hat{w}\right) + 2m\left(\lambda\rho J^1 + \lambda^{-1/2}\hat{w}\wedge\hat{\rho}\right)\right]. \tag{6.15}$$

The supersymmetry conditions (6.10)–(6.14) imply that the Killing vectors of the S^2, together with $\hat{w}$, are Killing vectors of the full solution that also preserve the flux. Thus supersymmetry implies that the generic isometry group is $\mathrm{SU}(2) \times \mathrm{U}(1)$. We are unaware of any explicit known solutions in this class.

7. Supersymmetric AdS_4 spacetimes

7.1 AdS_4 spacetimes from wrapping associative cycles

These geometries will be dual to three-dimensional SCFTs with minimal $\mathcal{N} = 1$ supersymmetry which, generically, have no R-symmetry.

For the associative wrapped brane geometries there is a single overall transverse direction so that $q = 1$. Imposing our AdS_4 limit we have $G' = \mathrm{SU}(3)$ with $e^7 = \hat{u}$ and thus

$$\mathrm{d}s^2(\mathcal{N}_7) = \mathrm{d}s^2(\mathcal{M}_{\mathrm{SU}(3)}) + \frac{\lambda^2\mathrm{d}\rho^2}{4m^2(1-\lambda^3\rho^2)}. \tag{7.1}$$

Defining the orientation

$$\epsilon = \tfrac{1}{6}J\wedge J\wedge J\wedge\hat{\rho}, \tag{7.2}$$

the supersymmetry conditions reduce to

$$\mathrm{d}\left(\lambda^{-3/2}\,\mathrm{Im}\,\Omega - \rho J\wedge\hat{\rho}\right) = 0, \tag{7.3}$$

$$\mathrm{d}\left(\frac{1}{\lambda\rho}J\wedge J + \frac{2}{\lambda^{5/2}\rho^2}\,\mathrm{Re}\,\Omega\wedge\hat{\rho}\right) = 0, \tag{7.4}$$

where the flux is given by

$$F = -\mathrm{d}\left(\frac{\sqrt{1-\lambda^3\rho^2}}{\lambda^{3/2}\rho}\,\mathrm{Re}\,\Omega\right) + m\left(\frac{1}{\lambda\rho}J\wedge J + \frac{2}{\lambda^{5/2}\rho^2}\,\mathrm{Re}\,\Omega\wedge\hat{\rho}\right). \tag{7.5}$$

In [4], Lukas and Saffin analysed the conditions for supersymmetry for a broad class of $\mathcal{N} = 1$ AdS_4 spacetimes with SU(3) structure in M-theory.[5] We have verified that the three equations for supersymmetry given above imply the conditions (3.40), (3.43)–(3.47) of [4]. However, our results imply expressions that are slightly different from equations (3.41) and (3.48) of [4].

The supersymmetry conditions in this case characterise the most general minimally supersymmetric AdS_4 spacetime in M-theory with purely magnetic fluxes. To see this, observe that we have derived them by taking the most general AdS limit of the associative calibration conditions of [2]. These in turn were obtained by setting the electric flux to zero in the conditions for the most general minimally supersymmetric $\mathbb{R}^{1,2}$ spacetime in

[5]The most general ansatz for the Killing spinors is given in [5].

M-theory, also given in [2]. The AdS_4 supersymmetry conditions in this case imply that generically the AdS limit of the associative calibration conditions has no isometries, which is consistent with lack of R symmetry in the dual SCFT. In section 9 we will recover the explicit solution of [9] using these results.

7.2 AdS_4 spacetimes from wrapping SLAG cycles in SU(3) holonomy

These geometries are dual to three-dimensional SCFTs with $\mathcal{N} = 2$ supersymmetry, which have a U(1) R-symmetry.

For the wrapped brane geometries corresponding to wrapping SLAG three-cycles there are two overall transverse directions so that $q = 2$. Imposing our AdS_4 limit we have $G' = \mathrm{SU}(2)$ in five dimensions with $e^6 = \hat{u}$. Writing $\hat{w} = e^5$ for the one-form used to define this structure, we have

$$\mathrm{d}s^2(\mathcal{N}_7) = \mathrm{d}s^2(\mathcal{M}_{\mathrm{SU}(2)}) + \hat{w} \otimes \hat{w} + \frac{\lambda^2}{4m^2}\left(\frac{\mathrm{d}\rho^2}{1-\lambda^3\rho^2} + \rho^2 \mathrm{d}\phi^2\right), \tag{7.6}$$

where ϕ is a coordinate on the S^1. Defining the orientation

$$\epsilon = \tfrac{1}{2} J^1 \wedge J^1 \wedge \hat{w} \wedge \hat{\rho} \wedge \widehat{\mathrm{vol}}(S^1), \tag{7.7}$$

the supersymmetry conditions reduce to

$$\begin{aligned}
\mathrm{d}[\lambda^{-1}\sqrt{1-\lambda^3\rho^2}\,\hat{w}] &= m\lambda^{-1/2} J^1 + m\lambda\rho\hat{w} \wedge \hat{\rho}, && (7.8)\\
\mathrm{d}\left(\lambda^{-3/2} J^3 \wedge \hat{w} - \rho J^2 \wedge \hat{\rho}\right) &= 0, && (7.9)\\
\mathrm{d}\left(J^2 \wedge \hat{w} + \lambda^{-3/2}\rho^{-1} J^3 \wedge \hat{\rho}\right) &= 0, && (7.10)
\end{aligned}$$

while the flux is given by

$$F = \frac{1}{\lambda\rho}\widehat{\mathrm{vol}}(S^1) \wedge \mathrm{d}\left(\lambda^{-1/2}\sqrt{1-\lambda^3\rho^2}\, J^3\right). \tag{7.11}$$

The supersymmetry conditions imply that $\partial/\partial\phi$ is a Killing vector of the full metric that also preserves the flux. Therefore the SLAG supersymmetry conditions imply that the AdS_4 limit generically has a U(1) isometry group, as expected from R-symmetry. In section 9 we will recover the explicit solution of [10] using these results.

8. Supersymmetric AdS_5 spacetimes

We now turn to the conditions for supersymmetric AdS_5 spacetimes obtained from supersymmetric wrapped-brane geometries. Two cases remain, corresponding to wrapping Kähler two-cycles in SU(3)- and SU(2)-holonomy manifolds. In fact, the first case is precisely the one considered in [6], where the reduction to AdS from a wrapped brane geometry was first discussed. This gave the conditions for the most generic supersymmetric AdS_5 spacetimes in M-theory dual to SCFTs with $\mathcal{N} = 1$ supersymmetry. We will not discuss this case any further but instead we turn directly to the case of wrapping Kähler two-cycles in SU(2)-holonomy manifolds, which preserves twice as much supersymmetry.

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8.1 AdS_5 spacetimes from wrapping Kähler two-cycles in SU(2) holonomy

These geometries are dual to four-dimensional SCFTs with $\mathcal{N} = 2$ supersymmetry, which have $\mathrm{SU}(2) \times \mathrm{U}(1)$ R-symmetry.

For the Kähler two-cycle in SU(2) holonomy wrapped brane geometries there are three overall transverse directions so that $q = 3$. Imposing our AdS_5 limit we find that the SU(2) structure is broken to a local identity structure in three dimensions defined by (e^1, e^2, e^3) with $e^4 = \hat{u}$, following the conventions of appendix A. We thus have

$$\mathrm{d}s^2(\mathcal{N}_6) = e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3 + \frac{\lambda^2}{4m^2}\left(\frac{\mathrm{d}\rho^2}{1-\lambda^3\rho^2} + \rho^2 \mathrm{d}\Omega_2^2\right). \tag{8.1}$$

Defining the orientation

$$\epsilon = e^{123} \wedge \hat{\rho} \wedge \widehat{\mathrm{vol}}(S^2), \tag{8.2}$$

the conditions for supersymmetry are

$$\begin{aligned}
\mathrm{d}\left(\lambda^{-1}\sqrt{1-\lambda^3\rho^2}e^1\right) &= m\lambda^{-1/2}\left(\lambda^{3/2}\rho e^1 \wedge \hat{\rho} + e^{23}\right), \\
\mathrm{d}\left(\lambda^{-1}\sqrt{1-\lambda^3\rho^2}e^2\right) &= m\lambda^{-1/2}\left(\lambda^{3/2}\rho e^2 \wedge \hat{\rho} - e^{13}\right), \\
\mathrm{d}\left(\frac{\lambda^{1/2}}{\sqrt{1-\lambda^3\rho^2}}e^3\right) &= -\frac{2m\lambda}{1-\lambda^3\rho^2}e^{12} - \frac{3\lambda\rho}{(1-\lambda^3\rho^2)^{3/2}}\left(\partial_{\hat{\rho}}\lambda e^{12}\right. \\
&\qquad \left. - \partial_2\lambda e^1 \wedge \hat{\rho} + \partial_1 \lambda e^2 \wedge \hat{\rho}\right),
\end{aligned} \tag{8.3}$$

and the flux is given by

$$F = -\frac{1}{\lambda^2\rho^2}\widehat{\mathrm{vol}}(S^2) \wedge \left[\mathrm{d}\left(\lambda^{1/2}\rho\sqrt{1-\lambda^3\rho^2}e^3\right) + 2m\left(\lambda\rho e^{12} + \lambda^{-1/2}e^3 \wedge \hat{\rho}\right)\right]. \tag{8.4}$$

These backgrounds preserve half of the supersymmetry. These conditions have in fact already been derived, from a somewhat different perspective, by Lin, Lunin and Maldacena (LLM) in [7]. As we show in detail in appendix D, our conditions are indeed equivalent to those of reference [7]. Specifically, we find that for the general solution of the conditions (8.3), we may take the metric to be given locally by

$$\begin{aligned}
\mathrm{d}s^2(\mathcal{N}_6) = &\frac{\lambda^2}{4m^2}\left[\frac{1}{1-\lambda^3\rho^3}(\mathrm{d}\rho^2 + \mathrm{e}^D \mathrm{d}x^i \mathrm{d}x^i) + \rho^2 \mathrm{d}\Omega_2^2\right] \\
&+ \frac{1-\lambda^3\rho^2}{\lambda m^2}\left(dx^3 + V_i dx^i\right)^2,
\end{aligned} \tag{8.5}$$

where $i = 1, 2$, the function $D(\rho, x^1, x^2)$ satisfies the Toda equation

$$\left(\partial_{x^1}^2 + \partial_{x^2}^2\right) D + \partial_\rho^2 \mathrm{e}^D = 0, \tag{8.6}$$

and the function λ and the one-form V are given by

$$\lambda^3 = -\frac{\partial_\rho D}{\rho(1-\rho\partial_\rho D)}, \tag{8.7}$$

$$V = \tfrac{1}{2} \star_2 \mathrm{d}_2 D, \tag{8.8}$$

where $\mathrm{d}_2 = \mathrm{d}x^i \, \partial_i$. The flux may be read off from (8.4). Note that here we have not assumed the $\mathrm{SU}(2) \times \mathrm{U}(1)$ isometry of these AdS spacetimes, as was done by LLM, but rather we have deduced it directly from the AdS limit of the supergravity description of the wrapped brane configuration.

9. Explicit solutions

In this section, we will discuss explicit solutions of the supersymmetry conditions for the AdS geometries we have just described. We have used two approaches, both of which reduce the problem to solving ordinary differential equations.

In the first approach one assumes that the metric on $\mathcal{M}_{G'}$ is conformal to a standard G'-structure metric: either a special holonomy metric or when $G' = \mathrm{SU}(3)$ a nearly Kähler metric. One then assumes that the conformal factor and the function λ depend only on the coordinate ρ.

The second approach is based on the class of known solutions originally derived using seven-dimensional gauged supergravity [8 − 11] and which describe M5-branes wrapping a variety of calibrated cycles. We start by identifying the relevant structures for these solutions. This will serve as a highly non-trivial consistency check on our general conditions as well as elucidating the geometrical structure underlying the solutions. In addition, this exercise suggests a natural class of generalisations, again depending only on ρ, and we derive the corresponding ordinary differential equations. In the case corresponding to wrapping a Kähler four-cycle in an SU(3)-holonomy manifold, we find some new, though singular, solutions.

We will begin this section by giving a general discussion of the G-structures of the known gauged supergravity AdS solutions. Then in the following subsections, we discuss the two approaches to finding more general solutions, in the cases of branes wrapping associative, co-associative or SLAG cycles, satisfying our general AdS supersymmetry conditions. In particular, we explicitly extract the G-structures underlying the gauged supergravity solutions in each case. We also include a subsection containing the new singular solutions for the AdS_3 limits in the case of a Kähler four-cycle in an SU(3)-holonomy manifold.

9.1 G-structures of gauged supergravity solutions

A general class of solutions [8 − 11] describing branes wrapping calibrated cycles in the near horizon limit, can be constructed by first finding AdS solutions in $D = 7$ gauged supergravity and then uplifting to $D = 11$. As such, the solutions all have the form of a warped product of $AdS_{7-d} \times \Sigma_d \times S^4$, where Σ_d is the cycle that the fivebrane is wrapping and the four-sphere surrounds the fivebrane. The four-sphere is fibred over Σ_d with the twisting determined by the structure of the normal bundle of a calibrated cycle in a special holonomy manifold.

More specifically, consider the solution for a fivebrane wrapping a calibrated Σ_d inside a special holonomy manifold. Following the discussion in [10], let p denote the number of dimensions transverse to the fivebrane worldvolume and tangent to the special holonomy

	p	q	a_1	a_2	$\mathrm{e}^{10\Lambda}$	c_1	c_2
SLAG in CY_3	3	2	$\mathrm{e}^{8\Lambda}$	$\mathrm{e}^{-2\Lambda}$	2	1	2
Kähler 2-cycle in CY_2	2	3	$2\mathrm{e}^{2\Lambda}$	$\mathrm{e}^{2\Lambda}$	2	1	2
Kähler 2-cycle in CY_3	4	1	$\frac{9}{4}\mathrm{e}^{4\Lambda}$	$\mathrm{e}^{-6\Lambda}$	$\frac{4}{3}$	$\frac{3}{2}$	2
associative in G_2-holonomy	4	1	$\frac{25}{16}\mathrm{e}^{4\Lambda}$	$\mathrm{e}^{-6\Lambda}$	$\frac{8}{5}$	$\frac{5}{4}$	2
co-associative in G_2-holonomy	3	2	$\frac{4}{9}\mathrm{e}^{8\Lambda}$	$\mathrm{e}^{-2\Lambda}$	3	$\frac{2}{3}$	2

Table 2: Examples of wrapped M5-brane solutions

manifold and q the number of dimensions transverse to both the fivebrane worldvolume and the special holonomy manifold. We thus have $p+q=5$ and the $\mathrm{SO}(5)$ symmetry of a flat fivebrane in flat space is broken to $\mathrm{SO}(p)\times\mathrm{SO}(q)$. For the known solutions the eleven-dimensional metric takes the form

$$\begin{aligned} m^2ds^2 = {} & \Delta^{-2/5}\left\{\frac{a_1}{u^2}\left[\mathrm{d}s^2(\mathbb{R}^{1,5-d})+\mathrm{d}u^2\right]+a_2\mathrm{d}s^2(\Sigma_d)\right\} \\ & +\Delta^{4/5}\left\{\mathrm{e}^{2q\Lambda}DY^aDY^a+\mathrm{e}^{-2p\Lambda}\mathrm{d}Y^\alpha\mathrm{d}Y^\alpha\right\}, \end{aligned} \tag{9.1}$$

where

$$\Delta^{-6/5}=e^{-2q\Lambda}(Y^aY^a+e^{10\Lambda}Y^\alpha Y^\alpha). \tag{9.2}$$

We have written the metric for AdS_{7-d} in Poincaré coordinates which displays the worldvolume of the fivebrane as $\mathbb{R}^{1,5-d}\times\Sigma_d$. The constants a_1 and a_2 specify the size of the AdS space and the cycle Σ. The coordinates Y^a, $a=1,\dots p$, and Y^α, $\alpha=1,\dots,q$, with $p+q=5$ parametrise the four-sphere: $Y^aY^a+Y^\alpha Y^\alpha=1$. We also have $DY^a=dY^a+B^a{}_bY^b$, where the $\mathrm{SO}(p)$ connection $B^a{}_b$ is determined by the spin connection of the cycle Σ_d. In particular, it is determined by the structure of the normal bundle to the calibrated cycle in the special holonomy manifold [8]. Furthermore, in the explicit solutions of [8 − 11] the cycles Σ_d are all Einstein, typically with negative curvature, and satisfy additional conditions that are discussed in the references.

The examples that are relevant for this paper are those with vanishing electric four-form components. We have listed the values of various quantities for these cases in table 2. Note that there is no entry for the Kähler 4 in CY_3 since there are no known solutions with AdS_3 factors.

To identify the underlying G-structure, it is illuminating to change coordinates from u, Y^a, Y^α to unconstrained "cartesian" coordinates X^a, X^α via

$$\begin{aligned} X^a &= u^{-c_1}Y^a, \qquad c_1=\mathrm{e}^{-2q\Lambda}\sqrt{a_1}, \\ X^\alpha &= u^{-c_2}Y^\alpha, \qquad c_2=\mathrm{e}^{2p\Lambda}\sqrt{a_1}, \end{aligned} \tag{9.3}$$

to obtain

$$\begin{aligned} m^2ds^2 = {} & \Delta^{-2/5}\left[\frac{a_1}{u^2}\mathrm{d}s^2(\mathbb{R}^{1,5-d})+a_2\mathrm{d}s^2(\Sigma_d)\right] \\ & +\Delta^{4/5}\left[\mathrm{e}^{2q\Lambda}u^{2c_1}DX^aDX^a+\mathrm{e}^{-2p\Lambda}u^{2c_2}\mathrm{d}X^\alpha\mathrm{d}X^\alpha\right], \end{aligned} \tag{9.4}$$

where $DX^a = dX^a + B^a{}_b X^b$. Although we have now obscured the AdS_{7-d} structure, the world-volume of the fivebrane, $\mathbb{R}^{1,5-d} \times \Sigma_d$, is still manifest and we have revealed[6] the $\mathbb{R}^p \times \mathbb{R}^q$ structure of the directions transverse to the fivebrane either tangent to the original special holonomy manifold ($\mathbb{R}^p$) or transverse to it ($\mathbb{R}^q$).

In this form it is straightforward to identify the structure that corresponds to the supersymmetry conditions that we discussed in section 3. Clearly $L = \Delta^{2/5} u^2/a_1$. We next note that for all of the cases considered in section 3, i.e. corresponding to wrapped brane solutions with no electric flux, we have $c_2 = 2$. As a consequence, after we rescale $X^\alpha \to \frac{1}{4} \mathrm{e}^{5p\Lambda} X^\alpha$ we see that the factor multiplying the overall transverse directions[7] is indeed L^2 in agreement with the discussion in section 3.

To display the rest of the structure in terms of the analysis of the wrapped-brane backgrounds, it is best to focus on an illustrative example. Consider the case of wrapping SLAG three-cycles in manifolds with SU(3)-holonomy. We have $p = 3$ corresponding to three directions transverse to the SLAG three-cycle inside the Calabi–Yau three-fold and $q = 2$ corresponding to the two overall transverse directions. For the solutions given in [28, 10], the cycle Σ_3 is hyperbolic space H_3 with the standard constant (unit) curvature metric, or a discrete quotient thereof, which may be compact. For this case the twisting is such that $B^a{}_b = \bar{\omega}^a{}_b$, the SO(3) spin connection of H_3. We now let $\bar{e}^a$ be an orthonormal frame for Σ_3 and consider the one forms $e^a = \Delta^{-1/5} \sqrt{a_2} m^{-1} \bar{e}^a$ and $f^a = \Delta^{2/5} \mathrm{e}^{q\Lambda} u^{c_1} m^{-1} DX^a$. Then, given the cycle is SLAG, the obvious SU(3) structure is

$$\begin{aligned} J &= e^a \wedge f^a, \\ \Omega &= \tfrac{1}{6} \epsilon^{abc} (e^a + \mathrm{i} f^a)(e^b + \mathrm{i} f^b)(e^c + \mathrm{i} f^c). \end{aligned} \tag{9.5}$$

One can check that the SLAG supersymmetry conditions of section 3.4 are indeed satisfied.

An advantage of displaying the structure at the level of the wrapped brane solutions is that the structures in the AdS limits are then easily identified, by carrying out the reduction procedure that we discussed in section 5. Returning to the general case, it is useful to introduce the following coordinates:

$$\begin{aligned} X^a &= u^{-c_1} \cos\tau \tilde{Y}^a, \\ X^\alpha &= u^{-c_2} \sin\tau \tilde{Y}^\alpha, \end{aligned}$$

where $\tilde{Y}^a$ parametrise a $(p-1)$-sphere, $\tilde{Y}^a \tilde{Y}^a = 1$, and $\tilde{Y}^\alpha$ parametrise a $(q-1)$-sphere, $\tilde{Y}^\alpha \tilde{Y}^\alpha = 1$. Obviously this is just equivalent to $Y^a = \cos\tau \tilde{Y}^a$ and $Y^\alpha = \sin\tau \tilde{Y}^\alpha$ in (9.3).

---

[6] The AdS solutions that we are discussing here are specific examples of more general solutions still with $\mathbb{R}^{1,5-d}$ and Σ_d factors but a more complicated dependence on the coordinate u which describe renormalisation group flows "across dimensions". The coordinate transformation we are describing can be generalised to this more general class of solutions. It was first noticed in the context of wrapped membranes [29].

[7] Interestingly $c_2 \neq 2$ for cases with non-zero electric flux which indicates that the factor multiplying the overall transverse directions will no longer be L^2. The most general minimally supersymmetric AdS_3 geometry with electric flux in M-theory is of this form, as discussed in [2].

We find that the metric now takes the form

$$\begin{aligned} m^2 \mathrm{d}s^2 = \Delta^{-2/5} & \left\{ \frac{a_1}{u^2} \left[\mathrm{d}s^2(\mathbb{R}^{1,5-d}) + \mathrm{d}u^2 \right] + a_2 ds^2(\Sigma_d) + \frac{\mathrm{e}^{2(q-p)\Lambda}}{a_1} \mathrm{d}\tau^2 \right\} \\ & + \Delta^{4/5} \left\{ \mathrm{e}^{2q\Lambda} \sin^2 \tau \mathrm{d}\tilde{Y}^\alpha \mathrm{d}\tilde{Y}^\alpha + \mathrm{e}^{2q\Lambda} \cos^2 \tau D\tilde{Y}^a D\tilde{Y}^a \right\}, \end{aligned} \tag{9.6}$$

and from the AdS factor we identify $\Delta^{-6/5} = (a_1\lambda)^{-3} = \mathrm{e}^{-2q\Lambda}\cos^2\tau + \mathrm{e}^{2p\Lambda}\sin^2\tau$. In order to make contact with the coordinates that we used in section 5, we introduce

$$\rho = 2a_1 \mathrm{e}^{-p\Lambda} \sin\tau. \tag{9.7}$$

We then find, for the cases with no electric flux that we are focussing on in this paper, that the metric becomes

$$\begin{aligned} m^2 \mathrm{d}s^2 = \lambda^{-1} & \left[\mathrm{d}s^2(AdS_{7-d}) + \frac{a_2}{a_1} \mathrm{d}s^2(\Sigma_d) + \frac{\mathrm{e}^{20\Lambda}}{4} (1 - \lambda^3\rho^2) D\tilde{Y}^a D\tilde{Y}^a \right] \\ & + \frac{\lambda^2}{4} \left(\frac{\mathrm{d}\rho^2}{1-\lambda^3\rho^2} + \rho^2 \mathrm{d}\tilde{Y}^\alpha \mathrm{d}\tilde{Y}^\alpha \right). \end{aligned} \tag{9.8}$$

This agrees with the general form (5.9) if we identify

$$\mathrm{d}s^2(\mathcal{M}_{G'}) = \frac{a_2}{a_1\lambda m^2} \mathrm{d}s^2(\Sigma_d) + \frac{\mathrm{e}^{20\Lambda}}{4\lambda m^2} (1-\lambda^3\rho^2) D\tilde{Y}^a D\tilde{Y}^a. \tag{9.9}$$

To identify the G-structure in the AdS limit, returning to the τ coordinate, we define the one-form

$$\begin{aligned} \hat{u} &= \frac{\Delta^{2/5}\mathrm{e}^{q\Lambda}}{m} \left(c_1 \cos\tau \frac{\mathrm{d}u}{u} + \sin\tau \mathrm{d}\theta \right) \\ &= \lambda^{-1/2}\sqrt{1-\lambda^3\rho^2}\mathrm{d}r + \frac{\lambda^{5/2}\rho \mathrm{d}\rho}{2m\sqrt{1-\lambda^3\rho^2}}, \end{aligned} \tag{9.10}$$

where we have defined $r = m^{-1}\log u$. This matches the unit one-form (5.4) introduced in section 5 which gives the component of the AdS radial direction in the space $\mathcal{M}_G$. We then have

$$\frac{\Delta^{2/5}\mathrm{e}^{q\Lambda}}{m} u^{c_1} DX^a = -\tilde{Y}^a \hat{u} + \frac{\Delta^{2/5}\mathrm{e}^{q\Lambda}}{m} \cos\tau D\tilde{Y}^a. \tag{9.11}$$

For the SLAG three-cycle case the two-form J of the SU(3) structure introduced above (9.5) now becomes

$$J = J^1 + \hat{w} \wedge \hat{u}, \tag{9.12}$$

where

$$\begin{aligned} \hat{w} &= -\frac{\sqrt{a_2}\Delta^{-1/5}}{m} \tilde{Y}^a \bar{e}^a, \\ J^1 &= \frac{\sqrt{a_2}\Delta^{1/5}\mathrm{e}^{q\Lambda}}{m^2} \cos\tau\, \bar{e}^a \wedge D\tilde{Y}^a. \end{aligned} \tag{9.13}$$

Furthermore analysing the expression for Ω, using its decomposition under SU(2) as given in appendix A, implies that

$$
\begin{aligned}
J^2 &= -\frac{\sqrt{a_2}\Delta^{1/5}\mathrm{e}^{q\Lambda}\cos\tau}{m^2}\epsilon^{abc}\tilde{Y}^a D\tilde{Y}^b \wedge \bar{e}^c, \\
J^3 &= \frac{\Delta^{4/5}\mathrm{e}^{2q\Lambda}\cos^2\tau}{2m^2}\epsilon^{abc}\tilde{Y}^a D\tilde{Y}^b \wedge D\tilde{Y}^c - \frac{a_2\Delta^{-2/5}}{2m^2}\epsilon^{abc}\tilde{Y}^a \bar{e}^b \wedge \bar{e}^c.
\end{aligned} \tag{9.14}
$$

The structures of the AdS solutions in the other cases may be similarly identified, both at the level of the wrapped brane structures and at the level of the structures in the AdS limits. In the next subsections we will use this to motivate some ansätze for solving the AdS supersymmetry conditions which include the explicit uplifted gauged supergravity solutions as special cases.

9.2 AdS_3 solutions from wrapping co-associative cycles

In this subsection, we will discuss three simple choices of the SU(3) structures which arise from the AdS_3 limit of the co-associative wrapped-brane spacetime, and which lead to simple solutions. Two are based on the first approach where we assume that the metric on $\mathcal{M}_{\mathrm{SU}(3)}$ is conformal to a standard geometry and the third follows from the structure of the known gauged supergravity solution.

$AdS_3 \times S^2 \times CY_3$ solutions The simplest family of solutions in the co-associative class is obtained by taking $\lambda =$ constant (up to an overall rescaling of the metric, we may choose $\lambda = 1$), and taking $\mathcal{M}_{\mathrm{SU}(3)}$ to be a Calabi–Yau, independent of the coordinates ρ and ϕ. Then it is immediately clear that equations (6.3) and (6.4) are satisfied, and the metric becomes the direct product $AdS_3 \times S^2 \times CY_3$. In fact, these are precisely the solutions (6.6) we found in the Kähler four-cycle in SU(3) holonomy class, when the AdS radial direction lay entirely in the overall transverse space. It is entirely consistent that they also arise here, since wrapping a Kähler four-cycle is a special case of wrapping a co-associative cycle in a G_2 holonomy manifold. Specifically we can write the G_2 structure metric in (3.1) as

$$
\mathrm{d}s^2(\mathcal{M}_{G_2}) = \mathrm{d}s^2(\mathcal{M}_{\mathrm{SU}(3)}) + L^2\mathrm{d}y_3^2, \tag{9.15}
$$

giving the same form as for the Kähler four-cycle wrapped brane geometry (3.8). Choosing the AdS radial direction to lie solely in the overall transverse space of the latter geometry thus corresponds to it lying partly in the overall transverse space and partly in $\mathcal{M}_{G_2}$ when viewed as a co-associative wrapped-brane geometry. From this perspective, the $AdS_3 \times S^2 \times CY_3$ solution to the Kähler four-cycle class is a special case of the co-associative AdS geometries, preserving twice as many supersymmetries.

Nearly-Kähler solutions A second family of solutions is obtained by assuming the metric on $\mathcal{M}_{\mathrm{SU}(3)}$ is not Calabi–Yau but is conformal to a nearly Kähler geometry. One takes

$$
\begin{aligned}
\lambda &= \lambda(\rho), \\
\mathrm{d}s^2(\mathcal{M}_{\mathrm{SU}(3)}) &= g^2(\rho)\,\mathrm{d}\tilde{s}^2(\mathcal{M}_{\mathrm{SU}(3)}),
\end{aligned} \tag{9.16}
$$

where the nearly Kähler (NK) metric $\mathrm{d}\tilde{s}^2$ is independent of ρ and ϕ. The NK structure implies that the rescaled local SU(3) structure given by $\tilde{J} = g^{-2}J$ and $\tilde{\Omega} = g^{-3}\Omega$ satisfies

$$\begin{aligned} \mathrm{d}\,\mathrm{Im}\,\tilde{\Omega} &= 0, \\ \mathrm{d}\,\mathrm{Re}\,\tilde{\Omega} &= c\tilde{J}\wedge\tilde{J}, \\ \mathrm{d}\tilde{J} &= \frac{3}{2}c\,\mathrm{Im}\,\tilde{\Omega}, \end{aligned} \tag{9.17}$$

with c a constant. In this case, equations (6.3) and (6.4) reduce to

$$\frac{\mathrm{d}}{\mathrm{d}\rho}(g^3) = -\frac{3cg^2}{4\lambda^{1/2}\rho m\sqrt{1-\lambda^3\rho^2}}, \tag{9.18}$$

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\Big(\frac{g^4}{\lambda}\Big) = -\frac{c\lambda^{3/2}\rho g^3}{m\sqrt{1-\lambda^3\rho^2}}. \tag{9.19}$$

Unfortunately, we have not found any explicit solutions of these equations. It may of course turn out to be the case that the general solution of these equations is singular, or that the metric of the general solution has the wrong signature (owing to the presence of the $1-\lambda^3\rho^2$ term).

Gauged supergravity inspired solutions Now let us recover and generalise the known gauged supergravity solution given in [10]. We make the metric ansatz

$$\begin{aligned} \lambda &= \lambda(\rho), \\ \mathrm{d}s^2(\mathcal{M}_{\mathrm{SU}(3)}) &= g^2(\rho)\mathrm{d}s^2(\Sigma_4) + f^2(\rho)D\tilde{Y}^aD\tilde{Y}^a, \end{aligned} \tag{9.20}$$

where the $\tilde{Y}^a$ are constrained coordinates on an S^2, $\tilde{Y}^a\tilde{Y}^a = 1$, and if J^a, $a = 1,2,3$ are a triplet of self-dual two-forms on Σ_4, taking the standard form (A.13), then

$$D\tilde{Y}^a = \mathrm{d}\tilde{Y}^a - \frac{1}{2}\epsilon^{abc}\tilde{Y}^b\omega_{ij}J^{cij}, \tag{9.21}$$

with $i,j = 1,\ldots,4$. Then we make the following ansatz for the SU(3) structure:

$$J = g^2\tilde{Y}^aJ^a + \frac{1}{2}f^2\epsilon^{abc}\tilde{Y}^aD\tilde{Y}^b\wedge D\tilde{Y}^c, \tag{9.22}$$

$$\mathrm{Im}\Omega = g^2fD\tilde{Y}^a\wedge J^a, \tag{9.23}$$

$$\mathrm{Re}\Omega = g^2f\epsilon^{abc}\tilde{Y}^aD\tilde{Y}^b\wedge J^c. \tag{9.24}$$

We begin by noting that $d(\tilde{Y}^aJ^a) = D\tilde{Y}^a\wedge J^a$ since it can be shown that $DJ^a = 0$. In addition we have $D^2\tilde{Y}^a = (1/4)\epsilon^{abc}J^{bij}R_{ijkl}e^k\wedge e^l\tilde{Y}^c$ where R_{ijkl} is the Riemann tensor of the metric on Σ_4. To demonstrate these facts it is useful to introduce a basis of anti-self-dual tensors K^a_{ij} satisfying $K^a_{ij}K^{bj}{}_k = -\delta^{ab}\delta_{ik} + \epsilon^{abc}K^c_{ik}$ and to note that $K^a_{ij}J^{bj}{}_k$ is a symmetric traceless tensor for each a,b. Furthermore, it is helpful to observe that $(1/2)J^{aij}J^a{}_{kl}$ is a projector onto self-dual tensors. It will also be useful to recall that in four dimensions the Riemann tensor can be decomposed as follows:

$$R_{ijkl} = C_{ijkl} + \delta_{i[k}\hat{R}_{l]j} - \delta_{j[k}\hat{R}_{l]i} + \frac{R}{6}\delta_{i[k}\delta_{l]j}, \tag{9.25}$$

where C_{ijkl} is the Weyl tensor, $\hat{R}_{ij}$ denotes the traceless part of the Ricci tensor,

$$R_{ij} = \hat{R}_{ij} + \frac{R}{4}\delta_{ij}, \tag{9.26}$$

with R the scalar Ricci curvature. In addition, we observe that the Weyl tensor may be expressed as

$$C_{ijkl} = A^{ab} J^a_{ij} J^b_{kl} + B^{ab} K^a_{ij} K^b_{kl}, \tag{9.27}$$

for some symmetric traceless A^{ab}, B^{ab}.

Using these results, if $\tilde{d}$ denotes the exterior derivative restricted to $\mathcal{M}_{\mathrm{SU}(3)}$, we then find

$$\begin{aligned} \tilde{d}J = & \Big(g^2 + \frac{Rf^2}{12}\Big) D\tilde{Y}^a \wedge J^a + \frac{f^2}{4} D\tilde{Y}^a \wedge J^{aij} C_{ijkl} e^k \wedge e^l \\ & + \frac{f^2}{2} D\tilde{Y}^a \wedge J^{aj}_{k} \hat{R}_{lj} e^k \wedge e^l, \end{aligned} \tag{9.28}$$

where C_{ijkl}, $\hat{R}_{ij}$ and R denote respectively the Weyl tensor, the traceless part of the Ricci tensor and the Ricci scalar of Σ_4. From equation (6.3) we find that we must have

$$J^{aij} C_{ijkl} = \hat{R}_{ij} = 0, \tag{9.29}$$

so Σ_4 must be conformally half-flat Einstein. Furthermore, it is readily verified that $\tilde{d}\mathrm{Im}\Omega = 0$. Equation (6.3) also gives the condition

$$\frac{d}{d\rho}(g^2 f) = -\frac{1}{2m\lambda^{1/2}\rho\sqrt{1-\lambda^3\rho^2}}\Big(g^2 + \frac{Rf^2}{12}\Big). \tag{9.30}$$

Given the conditions on the curvature of Σ_4, we find that

$$\tilde{d}\mathrm{Re}\Omega = \frac{g^2 f R}{3}\mathrm{Vol}(\Sigma_4) + g^2 f \tilde{Y}^d J^d \wedge \epsilon^{abc} \tilde{Y}^a D\tilde{Y}^b \wedge D\tilde{Y}^c. \tag{9.31}$$

In deriving the second term we found the following identity useful:

$$[\delta^{dc} - \tilde{Y}^d \tilde{Y}^c] \epsilon^{cab} D\tilde{Y}^a \wedge D\tilde{Y}^b = 0. \tag{9.32}$$

Then noting that $\tilde{d}(J \wedge J) = 0$, (6.4) gives the conditions

$$\frac{d}{d\rho}\Big(\frac{g^4}{\lambda}\Big) = -\frac{g^2 f \lambda^{3/2} \rho R}{6m\sqrt{1-\lambda^3\rho^2}}, \tag{9.33}$$

$$\frac{d}{d\rho}\Big(\frac{g^2 f^2}{\lambda}\Big) = -\frac{g^2 f \lambda^{3/2} \rho}{m\sqrt{1-\lambda^3\rho^2}}. \tag{9.34}$$

This pair of equations, together with the curvature conditions on Σ_4 and (9.30), are exhaustive for our ansatz. We observe that choosing $f = g$, $R = 6$, equations (9.30), (9.33) and (9.34) reduce to the equations for the nearly Kähler family discussed above, since then

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our ansatz together with the curvature conditions on Σ_4 implies that $\mathcal{M}_{\mathrm{SU}(3)}$ is nearly Kähler.

We have not found the general solution of (9.30), (9.33) and (9.34). However, it is readily verified that a particular solution is given by

$$\begin{aligned} R &= -4, \\ g^2 &= \frac{3}{4\lambda m^2}, \\ f^2 &= \frac{9(1-\lambda^3\rho^2)}{4\lambda m^2} \\ \lambda^3 &= \frac{3}{2(\rho^2+\alpha)}, \end{aligned} \tag{9.35}$$

for some constant α, which is essentially irrelevant as it must be positive for the metric to have the correct signature, and it may then be absorbed into an overall scale in the metric by rescaling ρ. It may be verified that choosing $\alpha = 32/27$ and defining constrained coordinates on an S^4, $Y^aY^a + Y^\alpha Y^\alpha = 1$, $\alpha = 4,5$, according to

$$Y^a = \sqrt{1-\frac{27}{64}\rho^2}\tilde{Y}^a, \tag{9.36}$$

$$Y^4 = \sqrt{\frac{27}{64}}\rho\sin\phi, \tag{9.37}$$

$$Y^5 = \sqrt{\frac{27}{64}}\rho\cos\phi, \tag{9.38}$$

we obtain precisely the metric (9.1), in the co-associative case, and hence the solution of [10].

9.3 AdS_3 spacetimes from wrapping Kähler four-cycles

Now we turn to the construction of two distinct (singular) families of AdS_3 spacetimes from Kähler-4 in CY_3 wrapped brane spacetimes, where the AdS radial direction lies partly in the CY_3 and partly in the overall transverse space. To the best of our knowledge, these are the first examples of this class of solutions to be constructed. (Recall that there are no gauged supergravity solutions in this class and that the $AdS_3 \times S^2 \times CY_3$ solution has the radial direction just in the overall transverse directions).

For the first family, we make the following ansatz

$$\lambda = \lambda(\rho), \tag{9.39}$$

$$\hat{w} = f(\rho)\tilde{w}(x), \tag{9.40}$$

$$ds^2(\mathcal{M}_{\mathrm{SU}(2)}) = g^2(\rho)\mathrm{d}\tilde{s}^2(\mathcal{M}_{\mathrm{SU}(2)}) \tag{9.41}$$

where $\mathrm{d}\tilde{s}^2$ is a ρ-independent metric of SU(2) holonomy. Equations (6.10) and (6.11) imply that

$$g^2 = \frac{\lambda^{1/2}}{\sqrt{1-\lambda^3\rho^2}}. \tag{9.42}$$

Equations (6.13) and (6.14) imply that

$$\hat{w} = \frac{\sqrt{1-\lambda^3\rho^2}}{\lambda^{1/2}}(d\psi + A(x)), \tag{9.43}$$

$$dA = 2m\alpha\tilde{J}^1 + K^{(-)}, \tag{9.44}$$

where α = constant, A is a one-form on $\mathcal{M}_{\mathrm{SU}(2)}$, $\tilde{J}^1 = g^{-2}J^1$, and $K^{(-)}$ denotes an arbitrary anti-self-dual two-form on $\mathcal{M}_{\mathrm{SU}(2)}$. Equation (6.12) then becomes

$$\frac{d}{d\rho}\Big(\frac{\lambda^{3/2}\rho}{\sqrt{1-\lambda^3\rho^2}}\Big)\tilde{J}^1 + \alpha\tilde{J}^1 + \frac{1}{2m}K^{(-)} = 0, \tag{9.45}$$

whence

$$K^{(-)} = 0, \tag{9.46}$$

$$\lambda^3 = \frac{(\beta-\alpha\rho)^2}{\rho^2[1+(\beta-\alpha\rho)^2]}, \tag{9.47}$$

for some constant β. We have now solved all the supersymmetry conditions, but it is readily verified that the resulting solution is singular for all values of α and β.

A second family of singular solutions may be obtained as follows. We make the ansatz

$$\lambda = \lambda(\rho), \tag{9.48}$$

$$e^a = g(\rho)dx^a, \quad a = 1,2, \tag{9.49}$$

$$e^p = h(\rho)dx^p \quad p = 3,4, \tag{9.50}$$

$$\hat{w} = f(\rho)(d\psi + A(x)), \tag{9.51}$$

$$dA = 2m\alpha dx^1 \wedge dx^2 + 2m\beta dx^3 \wedge dx^4, \tag{9.52}$$

where A is a one-form on $\mathcal{M}_{\mathrm{SU}(2)}$, and α and β are constants. Then equations (6.10) and (6.11) imply that

$$gh = \frac{\lambda^{1/2}}{\sqrt{1-\lambda^3\rho^2}}. \tag{9.53}$$

Equations (6.13) and (6.14) give

$$f = \frac{\sqrt{1-\lambda^3\rho^2}}{\lambda^{1/2}}. \tag{9.54}$$

The final condition for supersymmetry we must solve is (6.12), which reads

$$\frac{d}{d\rho}(\lambda\rho g^2) = -\alpha, \tag{9.55}$$

$$\frac{d}{d\rho}\Big(\frac{\lambda^2\rho}{(1-\lambda^3\rho^2)g^2}\Big) = -\beta. \tag{9.56}$$

Therefore

$$g^2 = \frac{\gamma-\alpha\rho}{\lambda\rho}, \tag{9.57}$$

$$\lambda^3 = \frac{(\gamma-\alpha\rho)(\delta-\beta\rho)}{\rho^2[1+(\gamma-\alpha\rho)(\delta-\beta\rho)]}, \tag{9.58}$$

for some constants γ, δ. It is readily verified that these solutions are singular for all values of $\alpha, \beta, \gamma, \delta$.

9.4 AdS_4 spacetimes from wrapping associative cycles

In this subsection, we will examine two distinct families of AdS_4 spacetimes arising from associative calibrations. As in the co-associative case, one of these families involves a nearly Kähler manifold, and we reduce the problem in this case to a pair of first-order ODEs, though we have not been able to find any explicit solutions. For the second family, we show that the problem reduces to the determination of three functions satisfying three first-order ODEs, which we show are satisfied by the explicit solution first constructed in gauged supergravity in [9].

NK_6 **solutions** To obtain the equations governing this family of solutions, we make the same metric ansatz as in the co-associative case, namely

$$\begin{aligned} \lambda &= \lambda(\rho), \\ \mathrm{d}s^2(\mathcal{M}_{\mathrm{SU}(3)}) &= g^2(\rho)\mathrm{d}\tilde{s}^2(\mathcal{M}_{\mathrm{SU}(3)}), \end{aligned} \tag{9.59}$$

with $\mathrm{d}\tilde{s}$ a ρ-independent nearly Kähler metric on $\mathcal{M}_{\mathrm{SU}(3)}$, so that it admits an SU(3) structure satisfying (9.17). Then equations (7.3) and (7.4) become

$$\frac{d}{d\rho}\Big(\frac{g^3}{\lambda^{3/2}}\Big) = -\frac{3c\lambda\rho g^2}{4m\sqrt{1-\lambda^3\rho^2}}, \tag{9.60}$$

$$\frac{d}{d\rho}\Big(\frac{g^4}{\lambda\rho}\Big) = -\frac{cg^3}{\lambda^{3/2}\rho^2 m\sqrt{1-\lambda^3\rho^2}}. \tag{9.61}$$

Unfortunately, we have not found any explicit solutions of these equations. This family of solutions has also been discussed in [4].

Gauged supergravity inspired solutions Let us now recover the explicit associative AdS_4 solution constructed in gauged supergravity in [9]. We make the metric ansatz

$$\begin{aligned} \lambda &= \lambda(\rho), \\ ds^2(\mathcal{M}_{\mathrm{SU}(3)}) &= f^2(\rho)\mu^a\mu^a + g^2(\rho)ds^2(\Sigma_3), \end{aligned} \tag{9.62}$$

where $a = 1, 2, 3$, e^a are a basis for Σ_3, and

$$\mu^a = \sigma^a - \frac{1}{2}\epsilon^{abc}\omega_{bc}, \tag{9.63}$$

where the σ^a are left-invariant one-forms on an S^3, $d\sigma^a = \frac{1}{2}\epsilon^{abc}\sigma^b \wedge \sigma^c$. We make the following ansatz for the SU(3) structure

$$J = fg\mu^a e^a, \tag{9.64}$$

$$\mathrm{Im}\Omega = \frac{1}{2}f^2 g\epsilon^{abc}e^a \wedge \mu^b \wedge \mu^c - \frac{1}{6}g^3\epsilon^{abc}e^a \wedge e^b \wedge e^c, \tag{9.65}$$

$$\mathrm{Re}\Omega = \frac{1}{6}f^3\epsilon^{abc}\mu^a \wedge \mu^b \wedge \mu^c - \frac{1}{2}fg^2\epsilon^{abc}\mu^a \wedge e^b \wedge e^c. \tag{9.66}$$

Denoting the exterior derivative restricted to $\mathcal{M}_{\mathrm{SU}(3)}$ by $\tilde{d}$, we find

$$\tilde{d}J = \frac{1}{2}fg\epsilon^{abc}\mu^a \wedge \mu^b \wedge e^c - \frac{1}{12}fgR\epsilon^{abc}e^a \wedge e^b \wedge e^c, \tag{9.67}$$

$$\tilde{d}\mathrm{Im}\Omega = 0, \tag{9.68}$$

$$\tilde{d}\mathrm{Re}\Omega = -f^3\hat{R}_{bc}e^a \wedge e^b \wedge \mu^a \wedge \mu^c - \Big(\frac{Rf^3}{12} + \frac{1}{2}fg^2\Big)e^a \wedge e^b \wedge \mu^a \wedge \mu^b, \tag{9.69}$$

where $\hat{R}_{ab}$ and R are respectively the traceless part of the Ricci tensor and the Ricci scalar of Σ_3. In three dimensions,

$$R_{abcd} = 2(\delta_{a[c}\hat{R}_{d]b} - \delta_{b[c}\hat{R}_{d]a}) + \frac{R}{3}\delta_{a[c}\delta_{d]b}. \tag{9.70}$$

We observe that if one sets $f = g$ and $R = 2$, one then gets a special case of the NK_6 family of solutions discussed above. In general, (7.3) and (7.4) imply that $\hat{R}_{ab} = 0$, R is constant and that

$$\frac{d}{d\rho}\Big(\frac{f^2g}{\lambda^{3/2}}\Big) = -\frac{\lambda\rho fg}{2m\sqrt{1-\lambda^3\rho^2}}, \tag{9.71}$$

$$\frac{d}{d\rho}\Big(\frac{g^3}{\lambda^{3/2}}\Big) = -\frac{\lambda\rho fgR}{4m\sqrt{1-\lambda^3\rho^2}}, \tag{9.72}$$

$$\frac{d}{d\rho}\Big(\frac{f^2g^2}{\lambda\rho}\Big) = -\frac{1}{\lambda^{3/2}\rho^2 m\sqrt{1-\lambda^3\rho^2}}\Big(\frac{Rf^3}{12} + \frac{1}{2}fg^2\Big). \tag{9.73}$$

We have not found the general solution of these equations. However, it is readily verified that a particular solution is given by

$$R = -6, \tag{9.74}$$

$$f^2 = \frac{4}{25\lambda m^2}(1 - \lambda^3\rho^2), \tag{9.75}$$

$$g^2 = \frac{4}{5\lambda m^2}, \tag{9.76}$$

$$\lambda^3 = \frac{8}{5}\frac{1}{(1+\frac{3}{5}\rho^2)}. \tag{9.77}$$

If we define a new coordinate θ such that

$$\rho = \sin\theta, \tag{9.78}$$

then up to an overall constant scale the metric is given by

$$ds^2 = 4\Delta^{1/3}X^8\Big[ds^2(AdS_4) + \frac{4}{5}ds^2(\Sigma_3)\Big] + X^3\Delta^{1/3}d\theta^2 + \frac{1}{4}\Delta^{-2/3}X^{-1}\cos^2\theta\mu^a\mu^a, \tag{9.79}$$

where

$$X = \Big(\frac{5}{8}\Big)^{1/5},$$
$$\Delta = X^{-4}\sin^2\theta + X\cos^2\theta. \tag{9.80}$$

This is exactly the eleven dimensional lift of the gauged supergravity solution given in section (3.2) of [9], in its original form (setting $h = 1$ in [9]). This may of course also be set in the form (9.1).

9.5 AdS_4 spacetimes from wrapping SLAG cycles

In this subsection, we will discuss a gauged supergravity inspired ansatz for the AdS_4 spacetime arising from M5-branes wrapping a SLAG three-cycle. In fact we will be able to solve all of the resulting equations and we will find that the gauged supergravity solution discussed in section 9.1 is the only regular one.

We make the following metric ansatz:

$$\lambda = \lambda(\rho), \tag{9.81}$$

$$\hat{w} \otimes \hat{w} + ds^2(\mathcal{M}_{\mathrm{SU}(2)}) = f^2(\rho) D\tilde{Y}^a D\tilde{Y}^a + g^2(\rho) ds^2(\Sigma_3), \tag{9.82}$$

where $a = 1, \dots, 3$, and again the $\tilde{Y}^a$ are constrained coordinates on an S^2, satisfying $\tilde{Y}^a \tilde{Y}^a = 1$. We define

$$D\tilde{Y}^a = d\tilde{Y}^a + \omega^a{}_b \tilde{Y}^b, \tag{9.83}$$

where ω_{ab} is the spin connection of Σ_3. We let e^a denote a basis for Σ_3, which we assume not to depend on the $\tilde{Y}^a$. Then we make the following ansatz for the structure:

$$\begin{aligned}
\hat{w} &= g \tilde{Y}^a e^a, && (9.84)\\
J^1 &= fg D\tilde{Y}^a \wedge e^a, && (9.85)\\
J^2 &= fg \epsilon^{abc} \tilde{Y}^a D\tilde{Y}^b \wedge e^c, && (9.86)\\
J^3 &= \frac{1}{2}\epsilon^{abc}[f^2 \tilde{Y}^a D\tilde{Y}^b \wedge D\tilde{Y}^c - g^2 \tilde{Y}^a e^b \wedge e^c]. && (9.87)
\end{aligned}$$

Note that we have flipped the signs of $\hat{w}$, J^1 and J^2 with respect to their definitions in subsection 9.1. We now insert this ansatz into (7.8), (7.9) and (7.10). Equation (7.8) immediately yields

$$\begin{aligned}
f &= \frac{\sqrt{1-\lambda^3\rho^2}}{\lambda^{1/2} m}, && (9.88)\\
\frac{d}{d\rho}\Big(\frac{\sqrt{1-\lambda^3\rho^2}\, g}{\lambda}\Big) &= -\frac{\lambda^2 \rho g}{2\sqrt{1-\lambda^3\rho^2}}. && (9.89)
\end{aligned}$$

The analysis of (7.9) and (7.10) is significantly more complicated, and it is helpful to use an equivalent form of these equations:

$$d\Big(\frac{1}{\lambda^{1/2}\sqrt{1-\lambda^3\rho^2}} J^3\Big) \wedge \hat{w} = -\frac{m\lambda^{3/2}\rho}{1-\lambda^3\rho^2} J^3 \wedge \hat{w} \wedge \hat{\rho} + \rho d\Big(\frac{\lambda}{\sqrt{1-\lambda^3\rho^2}} J^2\Big) \wedge \hat{\rho}, \tag{9.90}$$

$$d\Big(\frac{\lambda}{\sqrt{1-\lambda^3\rho^2}} J^2\Big) \wedge \hat{w} = -\frac{m\lambda^3\rho}{1-\lambda^3\rho^2} J^2 \wedge \hat{w} \wedge \hat{\rho} - \rho^{-1} d\Big(\frac{1}{\lambda^{1/2}\sqrt{1-\lambda^3\rho^2}} J^3\Big) \wedge \hat{\rho}. \tag{9.91}$$

We also note that $D^2\tilde{Y}^a = \frac{1}{2} R^a{}_{bcd} \tilde{Y}^b e^c \wedge e^d$. Now consider (9.91). It is straightforward to show that

$$\begin{aligned}
d\Big(\frac{\lambda}{\sqrt{1-\lambda^3\rho^2}} J^2\Big) \wedge \hat{w} = &-2m \frac{d}{d\rho}\log(\lambda^{1/2} g) J^2 \wedge \hat{w} \wedge \hat{\rho} \\
&+ \Big[(\delta^{ab} - \tilde{Y}^a \tilde{Y}^b)\omega_{abc} d\tilde{Y}^c + \epsilon^{abc}\epsilon^{def} \tilde{Y}^a \tilde{Y}^e d\tilde{Y}^b \omega_{fcd}\Big] \wedge Vol(\Sigma_3), \quad (9.92)
\end{aligned}$$

where

$$\epsilon^{abc} Vol(\Sigma_3) = e^a \wedge e^b \wedge e^c. \tag{9.93}$$

After some manipulation, the term is square brackets in (9.92) may be shown to vanish. Next, we find that

$$-\frac{1}{\rho} d\Big(\frac{1}{\lambda^{1/2}\sqrt{1-\lambda^3\rho^2}} J^3\Big) \wedge \hat{\rho} = \Big[-\frac{\sqrt{1-\lambda^3\rho^2}}{2\lambda^{3/2}\rho m^2} d(\epsilon^{abc}\tilde{Y}^a D\tilde{Y}^b \wedge D\tilde{Y}^c) + \frac{g^2}{2\lambda^{1/2}\rho\sqrt{1-\lambda^3\rho^2}} \epsilon^{abc} D\tilde{Y}^a \wedge e^b \wedge e^c \Big] \wedge \hat{\rho}. \tag{9.94}$$

Observing that

$$J^2 \wedge \hat{w} = \frac{fg^2}{2} \epsilon^{abc} D\tilde{Y}^a \wedge e^b \wedge e^c, \tag{9.95}$$

equation (9.91) becomes

$$\Big(2m\frac{d}{d\rho}\log(\lambda^{1/2}g) + \frac{m}{\rho}\Big) J^2 \wedge \hat{w} - \frac{\sqrt{1-\lambda^3\rho^2}}{2\lambda^{3/2}\rho m^2} d(\epsilon^{abc}\tilde{Y}^a D\tilde{Y}^b \wedge D\tilde{Y}^c) = 0. \tag{9.96}$$

It may be shown that

$$d(\epsilon^{abc}\tilde{Y}^a D\tilde{Y}^b \wedge D\tilde{Y}^c) = \epsilon^{abc}\tilde{Y}^a \tilde{Y}^d R_{bdef} D\tilde{Y}^c \wedge e^e \wedge e^f, \tag{9.97}$$

where R_{abcd} are the components of the Riemann tensor of Σ_3. Then (9.96) becomes

$$\Big(2m\frac{d}{d\rho}\log(\lambda^{1/2}g) + \frac{m}{\rho} + \frac{R}{6m\lambda\rho g^2}\Big) J^2 \wedge \hat{w} + \frac{1}{g\lambda\rho m} J^2 \wedge (\tilde{Y}^a \hat{R}_{ab} e^b) + \frac{\sqrt{1-\lambda^3\rho^2}}{g\lambda^{3/2}\rho m^2} \epsilon^{abc}\tilde{Y}^a D\tilde{Y}^b \wedge (\hat{R}_{cd} e^d) \wedge \hat{w} = 0. \tag{9.98}$$

Taking the wedge product of this equation with $\hat{w}$, we discover that

$$\tilde{Y}^a \hat{R}_{ab} d\tilde{Y}^b = 0, \tag{9.99}$$

and then taking the wedge product with J^2 we find that

$$\epsilon^{abc}\tilde{Y}^a D\tilde{Y}^b \wedge (\hat{R}_{cd} e^d) \wedge \hat{w} = 0. \tag{9.100}$$

Therefore

$$\hat{R}_{ab} = 0, \tag{9.101}$$

and thus Σ_3 is required to be Einstein, so it must be either H^3, S^3, or some quotient thereof. The remaining condition contained in (9.91) is

$$2\frac{d}{d\rho}\log(\lambda^{1/2}g) + \frac{1}{\rho} + \frac{R}{6m^2\lambda\rho g^2} = 0. \tag{9.102}$$

We may now obtain the general solution of (9.89), (9.102), and the full set of conditions we have derived hitherto may be summarised as follows.

$$\Sigma_3 \text{ is Einstein,} \tag{9.103}$$

$$f = \frac{\sqrt{1-\lambda^3\rho^2}}{m\lambda^{1/2}}, \tag{9.104}$$

$$g = \frac{1}{m\lambda^{1/2}}\left(\frac{\alpha}{\rho} - \frac{R}{6}\right)^{1/2}, \tag{9.105}$$

$$\lambda^3 = \frac{12\alpha - 2R\rho}{12\beta\rho - R\rho^3}, \tag{9.106}$$

for some constants α, β.

It remains to impose equation (9.90). After a similarly lengthy analysis, it may be shown that the only additional condition implied by this equation is

$$\alpha = 0. \tag{9.107}$$

Then for a real non-singular metric we must choose Σ_3 such that $R < 0$. The constant β is essentially irrelevant; in order for the metric to have the correct signature, we must take $\beta < 0$, and then by a constant rescaling of ρ it may be fixed to any particular value, up to an overall rescaling of the metric. Upon normalising $R = -3$, choosing β such that

$$\lambda^3 = \frac{1}{4(1+\rho^2/8)}, \tag{9.108}$$

and defining constrained coordinates on an S^4, Y^a, Y^α, $\alpha = 4, 5$, $Y^aY^a + Y^\alpha Y^\alpha = 1$, such that

$$Y^a = \sqrt{1-\frac{\rho^2}{8}}\tilde{Y}^a, \tag{9.109}$$

$$Y^4 = \frac{1}{2\sqrt{2}}\rho\sin\phi, \tag{9.110}$$

$$Y^5 = \frac{1}{2\sqrt{2}}\rho\cos\phi, \tag{9.111}$$

we obtain the metric (9.1), in the SLAG case. This is the eleven dimensional lift of the seven-dimensional solution originally found in [28].

10. Conclusions

In this paper we have given a general classification of supersymmetric geometries with AdS_{d+2} factors in M-theory in terms of G-structures. We have shown that the geometries can be obtained from an interesting class of spacetimes containing $\mathbb{R}^{1,d}$ factors and preserving algebraically the same set of Killing spinors as a probe M5-brane wrapping a calibrated cycle in a special holonomy manifold. We have also characterised this latter class of supersymmetric "wrapped-brane" spacetimes in terms of the corresponding G-structures.

The technique we have used for characterising the AdS geometries, by viewing them as special cases of Minkowski geometries of one dimension less, has allowed us to investigate

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numerous distinct classes in a way that is technically reasonably straightforward. Of course, the trade-off for this simplification is the loss of the guarantee of complete generality. However, in the case of AdS_5, it was shown in [6] that this approach does in fact lead to the most general supersymmetric AdS_5 geometries dual to $\mathcal{N}=1$ SCFTs. The work of [2], together with the results here, shows that this is also true for AdS_4 geometries with purely magnetic flux dual to $\mathcal{N}=1$ SCFTs. The work of [7], combined with our results, strongly suggests that it is true for AdS_5 geometries dual to $\mathcal{N}=2$ SCFTs, and we strongly suspect that it is also true for AdS_4 geometries with purely magnetic flux dual to $\mathcal{N}=2$ SCFTs. For AdS_3 geometries with vanishing electric flux that are dual to $\mathcal{N}=(2,0)$ supersymmetry it may also be true, though it may be, for example, that AdS_3 geometries arising from M5 branes wrapping Kähler four-cycles in Calabi-Yau four-folds with vanishing electric flux exist. To investigate this further one may well have to return to the standard approach of analysing the G-structure of the most general ansätze for the Killing spinors as in [6]. However, this will be complicated.

Another advantage, beyond technical tractability, of the techniques we have employed in this paper, is that by tracking the G-structure reduction induced by incorporating additional Killing spinors, we have been able to give a unified treatment of all the wrapped brane and AdS spacetimes we consider, by deriving the supersymmetry conditions in every case from the co-associative and associative calibration conditions. We have also seen how the R-symmetries of the dual SCFTs are encoded in the supergravity descriptions of the wrapped brane spacetimes, by elucidating how the isometries arise in the AdS limits.

It would be interesting to generalise the results here to cover other wrapped M5-brane geometries. For instance, it should be straightforward to extend our analysis to the case of M5-branes wrapping cycles in eight-dimensional manifolds, with $Spin(7)$, SU(4) or $Sp(2)$ holonomy. In these cases, electric charge can be induced from the Chern–Simons term in eleven-dimensional supergravity and so this should require a slight generalisation of the wrapped-brane ansatz to allow for this flux. The AdS_3 gauged supergravity solutions in [10, 11] corresponding to M5-branes wrapping calibrated cycles in eight-dimensional manifolds are of this type. This generalisation should allow for the classification of a variety of AdS_3 spacetimes with varying degrees of supersymmetry.

More generally, there are AdS_{d+1} geometries with electric flux which do not come from wrapped M5-branes, the simplest example being the basic Freund–Ruben AdS_4 solutions which are the near-horizon limit of a set of M2-branes at the apex of a cone with special holonomy contained in $Spin(7)$. Furthermore, an interesting example with dyonic fluxes is that of [30], where an AdS_4 solution with both electric and magnetic fluxes is constructed as the IR fixed point of a supersymmetric flow. Another natural generalisation, as in the previous paragraph, is thus to extend the wrapped-brane ansatz to include membrane probes or more generally dyonic probes which include both membrane and fivebrane charge. This kind of background appeared in the analysis of the generic minimally supersymmetric spacetimes with $\mathbb{R}^{1,2}$ given in [2].

An auxiliary result of our analysis is that all the supersymmetry conditions for the wrapped-brane spacetimes could be interpreted in terms of generalised calibrations and that this gave a relatively simply way of deriving the constraints on the geometry. A natural conjecture is that this is a general result. More precisely one might expect the conditions for supersymmetry for any given background to be equivalent to a set of eleven-dimensional generalised calibration conditions related to the allowed set of Killing spinors. In particular, the analysis of [15] implies that when the Killing spinor is timelike, the eleven-dimensional calibration conditions are indeed equivalent to the Killing spinor equation. Given an equivalent statement in the null case, the equivalence of supersymmetry conditions and the allowed set of generalised calibrations is then straightforward.

Compared to the success of [6] it has proved difficult to construct new explicit solutions. While we found some new examples, all were singular. However, in several cases we have reduced the problem of finding explicit solutions to that of solving a system of first-order ODEs. One might hope that a more in-depth (possibly numerical) analysis of these equations might lead to new solutions. And, of course, there is much scope for exploring further generalisations of the gauged supergravity solutions, which we leave to the future.

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A. Projections and structures

In this appendix we list a set of spinor projections which can be used to define the spinor ansätze for the wrapped-brane spacetimes. In all cases the spinors can be chosen to be eigenspinors of the five commuting projection operators

$$\left\{\Gamma^{1234}, \Gamma^{3456}, \Gamma^{5678}, \Gamma^{1357}, \Gamma^{+-}\right\}. \tag{A.1}$$

We will be interested in the cases of probe branes wrapping manifolds with G_2, SU(3) and SU(2) special holonomy.

Co-associative and associative calibrations in G_2 holonomy

We take the special holonomy geometry $\mathbb{R}^{1,3} \times \mathcal{M}_{G_2}$ with $\mathbb{R}^{1,3}$ spanned by $\{e^+, e^-, e^8, e^9\}$. One can define the four Killing spinors by

$$\Gamma^{1234}\epsilon^i = \Gamma^{3456}\epsilon^i = \Gamma^{1357}\epsilon^i = -\epsilon^i. \tag{A.2}$$

With this definition the G_2 structure takes the standard form

$$\begin{aligned} \Phi &= e^{127} + e^{347} + e^{567} + e^{246} - e^{136} - e^{145} - e^{235}, \\ \Upsilon &= e^{1234} + e^{3456} + e^{1256} + e^{1357} - e^{1467} - e^{2367} - e^{2457}. \end{aligned} \tag{A.3}$$

For a probe brane wrapping a co-associative cycle we have $d = 1$ in (2.1) and the unwrapped world-volume is spanned by $\{e^+, e^-\}$. We take the brane projection $\Gamma^{+-1234}\epsilon^i = -\epsilon^i$ or equivalently

$$\Gamma^{+-}\epsilon^i = \epsilon^i \qquad \text{(co-associative)}, \tag{A.4}$$

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and e^8 and e^9 define the overall transverse directions in $\mathcal{M}_9$.

For a probe brane wrapping an associative cycle we have $d = 2$ and the unwrapped worldvolume is spanned by $\{e^+, e^-, e^8\}$. We take the brane projection $\Gamma^{+-8567}\epsilon^i = \epsilon^i$ or equivalently

$$\Gamma^{+-}\Gamma^{5678}\epsilon^i = -\epsilon^i \qquad \text{(associative)}, \tag{A.5}$$

and e^9 defines the overall transverse direction in $\mathcal{M}_8$.

Kähler and SLAG calibrations in SU(3) holonomy

The special holonomy geometry is $\mathbb{R}^{1,4} \times \mathcal{M}_{\mathrm{SU}(3)}$ with $\mathbb{R}^{1,4}$ spanned by $\{e^+, e^-, e^7, e^8, e^9\}$. One can define the eight Killing spinors by

$$\Gamma^{1234}\epsilon^i = \Gamma^{3456}\epsilon^i = -\epsilon^i. \tag{A.6}$$

The SU(3) structure then takes the standard form

$$\begin{aligned} J &= e^{12} + e^{34} + e^{56}, \\ \Omega &= (e^1 + \mathrm{i}e^2) \wedge (e^3 + \mathrm{i}e^4) \wedge (e^5 + \mathrm{i}e^6). \end{aligned} \tag{A.7}$$

Further projecting under Γ^{1357} and comparing with (A.2) we see that this is equivalent to a pair of G_2 structures

$$\begin{aligned} \Phi_\pm &= \pm J \wedge e^7 - \operatorname{Im}\Omega, \\ \Upsilon_\pm &= \tfrac{1}{2} J \wedge J \pm \operatorname{Re}\Omega \wedge e^7. \end{aligned} \tag{A.8}$$

For a probe brane wrapping a Kähler four-cycle we have $d = 1$ and the unwrapped world-volume is spanned by $\{e^+, e^-\}$. We take the brane projection $\Gamma^{+-1234}\epsilon^i = -\epsilon^i$ or equivalently

$$\Gamma^{+-}\epsilon^i = \epsilon^i \qquad \text{(Kähler four-cycle)}, \tag{A.9}$$

and $\{e^7, e^8, e^9\}$ span the overall transverse directions in $\mathcal{M}_9$.

For a probe brane wrapping a Kähler two-cycle we have $d = 4$ and the unwrapped worldvolume is spanned by $\{e^+, e^-, e^7, e^8\}$. We take the brane projection $\Gamma^{+-7856}\epsilon^i = -\epsilon^i$ or equivalently

$$\Gamma^{+-}\Gamma^{5678}\epsilon^i = -\epsilon^i \qquad \text{(Kähler two-cycle)}, \tag{A.10}$$

and e^9 defines the overall transverse direction in $\mathcal{M}_7$.

For a probe brane wrapping a SLAG cycle we have $d = 3$ and the unwrapped worldvolume is spanned by $\{e^+, e^-, e^7\}$. We take the brane projection $\Gamma^{+-7135}\epsilon^i = \epsilon^i$ or equivalently

$$\Gamma^{+-}\Gamma^{1357}\epsilon^i = -\epsilon^i \qquad \text{(SLAG cycle)}, \tag{A.11}$$

and e^8 and e^9 define the overall transverse directions in $\mathcal{M}_8$.

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Kähler calibrations in SU(2) holonomy

The special holonomy geometry is $\mathbb{R}^{1,6} \times \mathcal{M}_{\mathrm{SU}(2)}$ with $\mathbb{R}^{1,6}$ spanned by $\{e^+, e^-, e^5, \ldots, e^9\}$. One can define the 16 Killing spinors by

$$\Gamma^{1234}\epsilon^i = -\epsilon^i. \tag{A.12}$$

The SU(2) structure then takes the standard form

$$\begin{aligned} J^1 &= e^{12} + e^{34}, \\ J^2 &= e^{14} + e^{23}, \\ J^3 &= e^{13} - e^{24}. \end{aligned} \tag{A.13}$$

Further projecting under Γ^{3456} and comparing with (A.6) we see this is equivalent to a pair of SU(3) structures

$$\begin{aligned} J &= J^1 \pm e^{56}, \\ \Omega &= (J^3 + \mathrm{i}J^2) \wedge (e^5 \pm \mathrm{i}e^6). \end{aligned} \tag{A.14}$$

For a probe brane wrapping a Kähler two-cycle we have $d = 4$ and the unwrapped worldvolume is spanned by $\{e^+, e^-, e^5, e^6\}$. We take the brane projection $\Gamma^{+-5634}\epsilon^i = -\epsilon^i$ or equivalently

$$\Gamma^{+-}\Gamma^{3456}\epsilon^i = -\epsilon^i \qquad \text{(Kähler two-cycle)}, \tag{A.15}$$

and $\{e^7, e^8, e^9\}$ span the overall transverse direction in $\mathcal{M}_7$.

B. AdS limits of wrapped brane metrics

In this appendix, we will give some further technical discussion of the assumptions we make in taking the AdS limit of the wrapped-brane metrics. Specifically, we will show that in the case of one overall transverse direction, the rotation angle θ must be independent of the AdS radial coordinate r, so in this case this requirement need not be imposed as an additional assumption. In the case of two or three overall transverse directions, we will show that with a suitable assumption of r-independence of the frame rotation, the part of the AdS radial direction lying in the overall transverse space must in fact lie entirely along the radial direction of the overall transverse space, as we assumed in the main text. We will discuss the cases of one, two, or three overall transverse directions in turn.

B.1 One overall transverse direction

There is one overall transverse direction for the cases of branes wrapping associative three-cycles or Kähler two-cycles in manifolds with $SU(3)$ holonomy. Then, necessarily, $\hat{v} = Ldt$. We want to show that the rotation angle θ must be independent of the AdS radial coordinate. We will see that this follows from the condition that the flux be independent of the AdS radial coordinate in the AdS limit, together with the fact that the flux for the wrapped brane metrics is completely determined by supersymmetry.

We will focus on proving this for the AdS limit of branes wrapping associative cycles; the argument for Kähler two-cycles in SU(3) holonomy is very similar. We have the relationships

$$\begin{aligned}\lambda^{-1/2}\mathrm{d}r &= \sin\theta\hat{u} + \cos\theta\hat{v},\\ \hat{\rho} &= \cos\theta\hat{u} - \sin\theta\hat{v},\\ ds^2(\mathcal{M}_{G_2}) &= ds^2(\mathcal{M}_{\mathrm{SU}(3)}) + \hat{u}\otimes\hat{u},\end{aligned} \tag{B.1}$$

where the metric in the AdS limit is

$$ds^2 = \lambda^{-1}ds^2(AdS_4) + ds^2(\mathcal{M}_{\mathrm{SU}(3)}) + \hat{\rho}\otimes\hat{\rho}. \tag{B.2}$$

By assumption, the metric on $\mathcal{M}_{\mathrm{SU}(3)}$ is independent of r. Therefore we may always choose the frame on $\mathcal{M}_{(\mathrm{SU}(3)}$ to be independent of r, which implies that $\hat{\rho}$ must be independent of r. Now, in the AdS limit, the expression (3.20) for the flux becomes

$$\begin{aligned}&\lambda e^{mr}\mathrm{d}\Big[\lambda^{-1}e^{-mr}\Big(\frac{1}{2}J\wedge J + \mathrm{Re}\Omega\wedge[\lambda^{-1/2}\sin\theta\mathrm{d}r + \cos\theta\hat{\rho}]\Big)\Big] =\\ &-(\cos\theta\lambda^{-1/2}\mathrm{d}r - \sin\theta\hat{\rho})\wedge F.\end{aligned} \tag{B.3}$$

Since F has no components along the AdS radial direction, we may read it off by comparing the dr terms on each side. In particular, we consider the components of F on $\mathcal{M}_{\mathrm{SU}(3)}$. These are given by

$$-\frac{m}{2\cos\theta}J\wedge J + \frac{\lambda}{\cos\theta}\tilde{\mathrm{d}}(\lambda^{-3/2}\sin\theta\mathrm{Re}\Omega), \tag{B.4}$$

where $\tilde{\mathrm{d}}$ denotes the exterior derivative restricted to $\mathcal{M}_{\mathrm{SU}(3)}$. The coefficient of the $J\wedge J$ part of this expression is

$$\frac{2\sin\theta\lambda^{-1/2}\mathrm{Re}\mathcal{W}_1 - m}{2\cos\theta}, \tag{B.5}$$

where we have used $\tilde{\mathrm{d}}\Omega = \mathcal{W}_1 J\wedge J + \ldots$. Since λ and $\mathcal{W}_1$ are independent of r, this flux component is independent of r iff θ is independent of r, as claimed. By a very similar argument, one may show that θ must also be independent of r for Kähler two-cycles in SU(3) holonomy manifolds.

B.2 Two overall transverse directions

Now we turn to the case of two overall transverse directions. We will use a slightly different set-up to that of the main text. We define the "wrapped brane frame"

$$\begin{aligned}e^1 &= Ldy^1,\\ e^2 &= Ldy^2,\\ e^3 &= \hat{u},\end{aligned} \tag{B.6}$$

where $y^{1,2}$ are cartesian coordinates on the overall transverse space. We define the "AdS frame"

$$(e^1)' = \lambda^{-1/2} dr, \tag{B.7}$$

with $(e^2)'$, $(e^3)'$ given by

$$(e^A)' = R_{AB} e^B, \tag{B.8}$$

for some $Spin(3)$ matrix R, with $A, B = 1, 2, 3$. Here we are viewing the AdS radial direction as arising from $e^{1,2,3}$ through what is *a priori* a completely general frame rotation. We wish to show that under the assumption that the matrix R is independent of the AdS radial coordinate r, we may always choose it such that

$$\begin{aligned}(e^2)' &= \hat{\rho}, \\ (e^3)' &= \frac{1}{2m}\lambda\rho \mathrm{d}\phi,\end{aligned} \tag{B.9}$$

with $\hat{\rho}$ as given in section 5. This is equivalent to the statement that assuming r-independence of R, the part of the AdS radial direction lying in the overall transverse space lies entirely along the radial direction of the overall transverse space.

In general, we have

$$dy^1 = \lambda^{-1} e^{-2mr} (R_{11}\lambda^{-1/2} dr + R_{21}(e^2)' + R_{31}(e^3)'), \tag{B.10}$$
$$dy^2 = \lambda^{-1} \mathrm{e}^{-2mr} (R_{12}\lambda^{-1/2} dr + R_{22}(e^2)' + R_{32}(e^3)'). \tag{B.11}$$

Now, given that R is independent of r, demanding that $dy^{1,2}$ are closed, and using the fact that R is a special orthogonal matrix, we find the following expressions for $(e^2)'$, $(e^3)'$:

$$(e^2)' = -\frac{\lambda}{2mR_{13}}(R_{32} d(\lambda^{-3/2} R_{11}) - R_{31} d(\lambda^{-3/2} R_{12})), \tag{B.12}$$
$$(e^3)' = \frac{\lambda}{2mR_{13}}(R_{22} d(\lambda^{-3/2} R_{11}) - R_{21} d(\lambda^{-3/2} R_{12})). \tag{B.13}$$

Next, defining coordinates ρ, ϕ such that

$$\lambda^{-3/2} R_{11} = \rho \cos\phi, \tag{B.14}$$
$$\lambda^{-3/2} R_{12} = \rho \sin\phi, \tag{B.15}$$

the $(e^2)'$, $(e^3)'$ become

$$(e^2)' = \frac{1}{2m\lambda^{1/2}}\Big(\frac{R_{23}}{\rho R_{13}} d\rho - R_{33} d\phi\Big), \tag{B.16}$$
$$(e^3)' = \frac{1}{2m\lambda^{1/2}}\Big(\frac{R_{33}}{\rho R_{13}} d\rho + R_{23} d\phi\Big). \tag{B.17}$$

We still have the freedom to perform rotations about the AdS radial direction, which we may exploit to choose a simpler frame. Thus, we perform a $Spin(2)$ rotation in the $2'3'$

plane, according to

$$
\begin{aligned}
(\hat{e}^2)' &= \frac{1}{\sqrt{1-R_{13}^2}}\Big(R_{23}(e^2)' + R_{33}(e^3)'\Big) = \frac{\lambda d\rho}{2m\sqrt{1-\lambda^3\rho^2}} = \hat{\rho}, \\
(\hat{e}^3)' &= \frac{1}{\sqrt{1-R_{13}^2}}\Big(-R_{33}(e^2)' + R_{23}(e^3)'\Big) = \frac{1}{2m}\lambda\rho d\phi,
\end{aligned} \tag{B.18}
$$

and so obtain the desired result.

B.3 Three overall transverse directions

The analysis with three overall transverse directions is qualitatively very similar to that with two, though it is technically somewhat more involved. We now take our "wrapped brane frame" to be given by

$$
\begin{aligned}
e^1 &= Ldy^1, \\
e^2 &= Ldy^2, \\
e^3 &= Ldy^3, \\
e^4 &= \hat{u}.
\end{aligned} \tag{B.19}
$$

Our "AdS frame" is given by

$$
(e^A)' = R_{AB}e^B, \tag{B.20}
$$

where now $A, B = 1, \ldots, 4$, R is a $Spin(4)$ matrix and $(e^1)' = \lambda^{-1/2}dr$. As before, we want to show that assuming that R is independent of the AdS radial coordinate r, we may always choose it such that

$$
\begin{aligned}
(e^2)' &= \hat{\rho}, \\
(e^3)' &= \frac{1}{2m}\lambda\rho \mathrm{d}\chi, \\
(e^4)' &= \frac{1}{2m}\lambda\rho\sin\chi \mathrm{d}\phi,
\end{aligned} \tag{B.21}
$$

and that therefore the part of the AdS radial direction lying in the overall transverse space must lie entirely along the radial direction of the overall transverse space.

In general, we have

$$
dy^1 = \lambda^{-1}e^{-2mr}(R_{11}\lambda^{-1/2}dr + R_{21}(e^2)' + R_{31}(e^3)' + R_{41}(e^4)'), \tag{B.22}
$$

$$
dy^2 = \lambda^{-1}\mathrm{e}^{-2mr}(R_{12}\lambda^{-1/2}dr + R_{22}(e^2)' + R_{32}(e^3)' + R_{42}(e^4)'), \tag{B.23}
$$

$$
dy^3 = \lambda^{-1}\mathrm{e}^{-2mr}(R_{13}\lambda^{-1/2}dr + R_{23}(e^2)' + R_{33}(e^3)' + R_{43}(e^4)'). \tag{B.24}
$$

Now, given that R is independent of r, demanding that $dy^{1,2,3}$ are closed, we get

$$
(e^2)' = \frac{\lambda}{2mR_{14}}\epsilon^{ijk4}R_{3i}R_{4j}d(\lambda^{-3/2}R_{1k}), \tag{B.25}
$$

$$
(e^3)' = \frac{\lambda}{2mR_{14}}\epsilon^{ijk4}R_{4i}R_{2j}d(\lambda^{-3/2}R_{1k}), \tag{B.26}
$$

$$
(e^4)' = \frac{\lambda}{2mR_{14}}\epsilon^{ijk4}R_{2i}R_{3j}d(\lambda^{-3/2}R_{1k}), \tag{B.27}
$$

where $i,j,k = 1,\ldots,4$ and $\epsilon^{1234} = 1$. We still have the freedom to perform $Spin(3)$ rotations about the AdS radial direction, to simplify the frame. To this end, we define coordinates ρ, χ, ϕ such that

$$\lambda^{-3/2} R_{11} = \rho \sin\chi \sin\phi, \tag{B.28}$$

$$\lambda^{-3/2} R_{12} = \rho \sin\chi \cos\phi, \tag{B.29}$$

$$\lambda^{-3/2} R_{13} = \rho \cos\chi. \tag{B.30}$$

With this choice of coordinates, our frame is given by

$$\begin{pmatrix} (e^2)' \\ (e^3)' \\ (e^4)' \end{pmatrix} = \frac{Q}{2m} \begin{pmatrix} \frac{\lambda}{\sqrt{1-\lambda^3\rho^2}} d\rho \\ \lambda\rho d\chi \\ \lambda\rho \sin\chi d\phi \end{pmatrix}, \tag{B.31}$$

where the matrix Q is given by

$$Q = \frac{1}{\lambda^{3/2}\rho} \begin{pmatrix} R_{24} & \frac{1}{\lambda^{3/2}\rho\sin\chi R_{14}}(R_{24}R_{13} - \lambda^3\rho^2\epsilon^{34ij}R_{3i}R_{4j}) & \frac{1}{\sin\chi}\epsilon^{12ij}R_{3i}R_{4j} \\ R_{34} & \frac{1}{\lambda^{3/2}\rho\sin\chi R_{14}}(R_{34}R_{13} - \lambda^3\rho^2\epsilon^{34ij}R_{4i}R_{2j}) & \frac{1}{\sin\chi}\epsilon^{12ij}R_{4i}R_{2j} \\ R_{44} & \frac{1}{\lambda^{3/2}\rho\sin\chi R_{14}}(R_{44}R_{13} - \lambda^3\rho^2\epsilon^{34ij}R_{2i}R_{3j}) & \frac{1}{\sin\chi}\epsilon^{12ij}R_{2i}R_{3j} \end{pmatrix}. \tag{B.32}$$

It may be verified that Q is an element of $Spin(3)$. Therefore we may rotate about the AdS radial direction to get a new frame, given by

$$\begin{pmatrix} (\hat{e}^2)' \\ (\hat{e}^3)' \\ (\hat{e}^4)' \end{pmatrix} = Q^{-1} \begin{pmatrix} (e^2)' \\ (e^3)' \\ (e^4)' \end{pmatrix} = \frac{1}{2m} \begin{pmatrix} \frac{\lambda}{\sqrt{1-\lambda^3\rho^2}} d\rho \\ \lambda\rho d\chi \\ \lambda\rho \sin\chi d\phi \end{pmatrix}, \tag{B.33}$$

as required.

C. Sample calculations of the supersymmetry conditions

In this appendix, we will give more details of a representative example of the derivation of the AdS supersymmetry conditions from the wrapped brane supersymmetry conditions. We will focus on the derivation of the $\mathcal{N} = 2$ AdS_4 supersymmetry conditions from the SLAG supersymmetry conditions. We have the following expressions for the basis one-forms in the "wrapped brane frame" in terms of the coordinates in the "AdS frame":

$$\hat{u} = \lambda^{-1/2}\sqrt{1-\lambda^3\rho^2}dr + \frac{\lambda^{5/2}\rho d\rho}{2m\sqrt{1-\lambda^3\rho^2}}, \tag{C.1}$$

$$Ldt = \lambda\rho dr - \frac{\lambda}{2m} d\rho, \tag{C.2}$$

$$Ltd\phi = -\frac{\lambda\rho}{2m} d\phi. \tag{C.3}$$

The SU(3) forms appearing in (3.23)-(3.26) decompose into SU(2) forms according to

$$\begin{aligned} J &= J^1 + \hat{w}\wedge\hat{u}, \\ \mathrm{Re}\Omega &= J^3\wedge\hat{w} - J^2\wedge\hat{u}, \\ \mathrm{Im}\Omega &= J^2\wedge\hat{w} + J^3\wedge\hat{u}, \end{aligned} \tag{C.4}$$

where the J^i are given by (A.13). We define the new frame

$$\hat{\rho} = \frac{\lambda}{2m\sqrt{1-\lambda^3\rho^2}} d\rho, \tag{C.5}$$

$$\hat{\phi} = \frac{\lambda\rho}{2m} d\phi, \tag{C.6}$$

$$\hat{r} = \lambda^{-1/2} dr. \tag{C.7}$$

The frame in the directions transverse to the AdS factor is independent of r. Equation (3.24) becomes

$$d\Big[\lambda^{-1/2} \mathrm{e}^{-mr} \Big(J^1 + \hat{w} \wedge (\lambda^{-1/2}\sqrt{1-\lambda^3\rho^2} dr + \frac{\lambda^{5/2}\rho}{2m\sqrt{1-\lambda^3\rho^2}} d\rho)\Big)\Big] = 0, \tag{C.8}$$

which reads

$$d[\lambda^{-1}\sqrt{1-\lambda^3\rho^2}\hat{w}] = m(\lambda^{-1/2} J^1 + \lambda\rho\hat{w} \wedge \hat{\rho}), \tag{C.9}$$

which is (7.8). Next, imposing $\hat{r} \lrcorner F = 0$, we find that (3.26) becomes

$$d[\lambda^{-3/2} J^3 \wedge \hat{w} - \rho J^2 \wedge \hat{\rho}] = 0, \tag{C.10}$$

$$\hat{r} \wedge \lambda^2 \Big[d[\lambda^{-2}\sqrt{1-\lambda^3\rho^2} J^2] - 3m(\lambda^{-3/2} J^3 \wedge \hat{w} - \rho J^2 \wedge \hat{\rho})\Big] = \star_8 F. \tag{C.11}$$

The first of these equations is (7.9). Next, (3.25) becomes

$$d\phi \wedge d[J^2 \wedge \hat{w} + \frac{1}{\lambda^{3/2}\rho} J^3 \wedge \hat{\rho}] = 0. \tag{C.12}$$

This is consistent with (7.10) but does not imply it. However, observe that (C.9) implies that

$$\partial_\phi(\lambda^{-1}\sqrt{1-\lambda^3\rho^2}\hat{w}) = 0, \tag{C.13}$$

and the exterior derivative of (C.9) implies that

$$\partial_\phi(\lambda^{-1/2} J^1) = \partial_\phi\Big(\frac{\lambda^2}{\sqrt{1-\lambda^3\rho^2}}\hat{w}\Big) = 0, \tag{C.14}$$

and therefore that

$$\partial_\phi\lambda = \partial_\phi\hat{w} = \partial_\phi J^1 = 0. \tag{C.15}$$

Then (C.10) implies that

$$\partial_\phi J^2 = \partial_\phi J^3 = 0, \tag{C.16}$$

and hence (C.9), (C.10) and (C.12) imply equation (7.10). Furthermore, (C.15) and (C.16) imply that ∂_ϕ is Killing, as claimed in the main text.

The final SLAG torsion condition is $\mathrm{Re}\Omega \wedge d\mathrm{Re}\Omega = 0$. Rewriting this as

$$\lambda^{-3/2}\mathrm{Re}\Omega \wedge d(\lambda^{-3/2}\mathrm{Re}\Omega) = 0, \tag{C.17}$$

and using (C.10), we get

$$\begin{aligned}(\lambda^{-3/2}J^3 \wedge \hat{w} - \rho J^2 \hat{\rho} \wedge d(\lambda^{-2}\sqrt{1-\lambda^3\rho^2}J^2) &= \frac{1}{2m\rho} d(\lambda^{-1}J^2 \wedge J^2) \wedge d\rho \\ &= \frac{1}{2m\rho} d(\lambda^{-1}J^1 \wedge J^1) \wedge d\rho = 0. \end{aligned} \tag{C.18}$$

But this is automatically satisfied as a consequence of the exterior derivative of (C.9), which states that

$$d(\lambda^{-1/2}J^1) = -\frac{3\rho}{1-\lambda^3\rho^2} d\lambda \wedge \hat{w} \wedge \hat{\rho} - \frac{m\lambda^{3/2}\rho}{\sqrt{1-\lambda^3\rho^2}} J^1 \wedge \hat{\rho}. \tag{C.19}$$

It remains to obtain the expression (7.11) for the flux. To do this, we use (3.27), with $v = \hat{v} = Ldt$. We get

$$\hat{\phi} \wedge d\Big[J^2 \wedge \hat{w} + J^3 \wedge \Big(\lambda^{-1/2}\sqrt{1-\lambda^3\rho^2}dr + \frac{\lambda^{5/2}\rho}{2m\sqrt{1-\lambda^3\rho^2}}d\rho\Big)\Big] = (\lambda\rho dr - \frac{\lambda}{2m}d\rho) \wedge F. \tag{C.20}$$

Since F has no components along the AdS radial direction, we may simply read it off directly by comparing the dr terms on both sides. It is simple to see that the Killing vector ∂_ϕ leaves the flux invariant.

D. Deriving the LLM conditions

In this appendix, we will show that the general solution of our $\mathcal{N} = 2$ AdS_5 supersymmetry conditions precisely satisfies the conditions derived by LLM [7]. To begin, let us define

$$e^1 = \frac{\lambda}{2m\sqrt{1-\lambda^3\rho^2}} \tilde{e}^1, \tag{D.1}$$

$$e^2 = \frac{\lambda}{2m\sqrt{1-\lambda^3\rho^2}} \tilde{e}^2, \tag{D.2}$$

$$e^3 = \frac{\sqrt{1-\lambda^3\rho^2}}{m\lambda^{1/2}} \tilde{e}^3. \tag{D.3}$$

Then equations (8.3) become

$$d\tilde{e}^1 = -\frac{\lambda^3\rho}{2(1-\lambda^3\rho^2)} d\rho \wedge \tilde{e}^1 + \tilde{e}^{23}, \tag{D.4}$$

$$d\tilde{e}^2 = -\frac{\lambda^3\rho}{2(1-\lambda^3\rho^2)} d\rho \wedge \tilde{e}^2 + \tilde{e}^{31}, \tag{D.5}$$

$$\begin{aligned} 2d\tilde{e}^3 &= -\frac{\lambda^3}{(1-\lambda^3\rho^2)^2}\tilde{e}^{12} - \frac{\rho}{(1-\lambda^3\rho^2)^2}\Big(\partial_\rho \lambda^3 \tilde{e}^{12} \\ &\quad - [\tilde{\partial}_2\lambda^3\tilde{e}^1 - \tilde{\partial}_1\lambda^3\tilde{e}^2] \wedge d\rho\Big). \end{aligned} \tag{D.6}$$

Equation (D.4) implies that we may write

$$\tilde{e}^1 = \mathrm{e}^{\frac{1}{2}D(\rho,x^a)}\hat{e}^1(x^a), \tag{D.7}$$

where the x^a, $a = 1, 2, 3$ are some coordinates on the three-space spanned by the $\tilde{e}^a$ (which we refer to as the base), and furthermore that

$$\lambda^3 = -\frac{\partial_\rho D}{\rho(1-\rho\partial_\rho D)}. \tag{D.8}$$

Similarly from (D.5), we find that we may write

$$\tilde{e}^2 = \mathrm{e}^{\frac{1}{2}D}\hat{e}^2(x^a). \tag{D.9}$$

Then if we denote the exterior derivative restricted to the base by $\tilde{d}$, the remaining content of (D.4), (D.5) is

$$\tilde{d}\hat{e}^1 = -\frac{1}{2}\tilde{d}D \wedge \hat{e}^1 + \hat{e}^2 \wedge \tilde{e}^3, \tag{D.10}$$

$$\tilde{d}\hat{e}^2 = -\frac{1}{2}\tilde{d}D \wedge \hat{e}^2 - \hat{e}^1 \wedge \tilde{e}^3. \tag{D.11}$$

Next, from (D.6), we find that

$$(\partial_\rho \tilde{e}^3)_3 = 0. \tag{D.12}$$

Therefore we may choose our coordinates such that as a vector

$$\tilde{e}^3 = \frac{\partial}{\partial x^3}, \tag{D.13}$$

and as a one-form

$$\tilde{e}^3 = (dx^3 + V_{\hat{i}}(\rho, x^a)\hat{e}^i), \tag{D.14}$$

where $\hat{i} = 1, 2$. Now, by taking the ρ derivative of (D.10) and (D.11), we find that

$$\partial_{\rho x^3} D = 0, \tag{D.15}$$

$$\partial_\rho(\hat{\partial}_2 D - 2V_{\hat{1}}) = 0, \tag{D.16}$$

$$\partial_\rho(\hat{\partial}_1 D + 2V_{\hat{2}}) = 0. \tag{D.17}$$

We are free to shift the definition of D by an arbitrary function of the x^a. Thus (D.15) implies that we may always take

$$D = D(\rho, x^1, x^2). \tag{D.18}$$

Then the x^3 dependence of $\hat{e}^1$, $\hat{e}^2$ is fixed by (D.10) and (D.11) to be given by

$$\partial_{x^3}\hat{e}^1 = -\hat{e}^2, \tag{D.19}$$

$$\partial_{x^3}\hat{e}^2 = \hat{e}^1. \tag{D.20}$$

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Therefore, we have

$$\hat{e}^1 = \sin x^3 \overline{e}^1(x^1, x^2) + \cos x^3 \overline{e}^2(x^1, x^2), \tag{D.21}$$
$$\hat{e}^2 = -\cos x^3 \overline{e}^1(x^1, x^2) + \sin x^3 \overline{e}^2(x^1, x^2). \tag{D.22}$$

The absence of a term with $\tilde{e}^3$ on the r.h.s. of (D.6) implies that

$$V_{\hat{i}} = V_{\hat{i}}(\rho, x^1, x^2). \tag{D.23}$$

Returning to equations (D.16) and (D.17), and denoting the exterior derivative restricted to the two-space spanned by the $\overline{e}^i$ by $\overline{d}$, we see that we may write

$$V = \frac{1}{2} \star_2 \overline{d}D + \xi(x^a), \tag{D.24}$$

for some one-form ξ. Since V and D are independent of x^3, then so also is ξ. We still have two gauge degrees of freedom left, which we may use to set $\xi = 0$. To see this, observe that we may shift D by an arbitrary function of x^i, $D \to D + 2f(x^1, x^2)$, and by means of a shift in x^3, we may set $V \to V + dg$, for some arbitrary function $g(x^1, x^2)$. Thus we may always take $\xi = 0$ if we can solve

$$dg = \star_2 df + \xi. \tag{D.25}$$

Taking the exterior derivative of this equation and its dual, we find that we may set $\xi = 0$ if we can find functions f, g that solve

$$\begin{aligned} d \star_2 df &= -d\xi, \\ d \star_2 dg &= d \star_2 \xi. \end{aligned} \tag{D.26}$$

But these are just two independent copies of Poisson's equation in two Riemannian dimensions, and we may always find a solution of each in a local coordinate patch. Therefore we may always take $V = \frac{1}{2} \star_2 \overline{d}D$, and (D.10) and (D.11) reduce to

$$d\overline{e}^i = 0, \tag{D.27}$$

for which we take the local solution

$$\overline{e}^i = dx^i. \tag{D.28}$$

It may now be verified that upon inserting all the conditions we have derived above, equation (D.6) reduces to the Toda equation

$$(\partial^2_{x^1} + \partial^2_{x^2})D + \partial^2_\rho \mathrm{e}^D = 0. \tag{D.29}$$

Given a solution of this equation, the metric is given by

$$\begin{aligned} ds^2 = \frac{1}{\lambda m^2}\Big[ds^2(AdS_5) + \frac{\lambda^3}{4}\Big(\frac{1}{1-\lambda^3\rho^3}(d\rho^2 + \mathrm{e}^D dx^i dx^i) + \rho^2 ds^2(S^2)\Big) \\ +(1-\lambda^3\rho^2)(dx^3 + V_i dx^i)^2\Big], \end{aligned} \tag{D.30}$$

where

$$\lambda^3 = -\frac{\partial_\rho D}{\rho(1-\rho\partial_\rho D)}, \tag{D.31}$$

$$V = \frac{1}{2} \star_2 \overline{d}D, \tag{D.32}$$

and the flux may be read off from (8.4). As claimed in the main text, these are precisely the LLM conditions, which are given in [7] for $m = 1/2$.

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